General Identities for Parameterizations of SO(3) with Applications

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ABSTRACT

Rotation matrices, which are three-by-three orthonormal matrices with determinant equal to plus one, constitute the special orthogonal group of rigid-body rotations, denoted SO(3). Owing to the three-by-three nature of rotation matrices plus their orthonormality constraint, parameterizations are often used in favour of rotation matrices for computations and derivations. For example, Euler angles and Rodrigues parameters are common three-parameter unconstrained parameterizations, while unit-length quaternions are a popular four-parameter constrained parameterization. In this paper various identities associated with the parameterization of SO(3) are considered. In particular, we present six identities, three related to unconstrained parameterizations, and three related to constrained parameterizations. We also discuss rotation matrix perturbations. The utility of these identities is highlighted when deriving the motion equations of a rigid body using Lagrange's equation. We also use them to examine some issues associated with spacecraft attitude determination.

1 Introduction

The special orthogonal group of rigid body rotations, denoted SO(3), consists of all three-by-three matrices that realize rotational transformations between dextral reference frames. While these matrices consist of nine elements, they also have six constraints, thus realizing three degrees of freedom [1]. Recently there has been interest in working with rotation matrices directly, especially for control and estimation; for example, see the recent survey in [2]. The primary benefit of working directly on SO(3) is that attitude representations are global (hence singularity free) and unique.

Although it is theoretically elegant to work with rotation matrices directly, the fact that rotation matrices have nine elements that must satisfy the orthonormality constraint makes them difficult to work with in practice. For example, direct numerical integration of Poisson's equation (which describes the time-evolution of a rotation matrix) will eventually lead to violation of the orthonormality constraint. As such, in practice parameterizations of SO(3) are often employed. For example, Euler angles, Rodrigues parameters, and unit-length quaternions (often called quaternions for simplicity) are popular parameterizations used in many applications, such as dynamics, control, and estimation [3]; see [4] for a survey of different parameterizations. Each of these parameterizations has their own benefits and disadvantages.

Dynamics, control, and estimation problems often have many degrees of freedom. Although it is possible to, for example, derive the equations of motion of a multibody system in scalar form and then place the scalar equations in matrix-vector form, doing so is tedious and prone to error. The motivation of the present work is the development of identities and perturbation techniques specific to attitude parameterizations that simplify and streamline various computations found in dynamics, control, and estimation.

The novel contributions of this paper are three fold. First, six identities related to the time-rate-of-change and the partial derivatives of both constrained and unconstrained parameterizations of SO(3) are rigorously proven. These identities are particularly useful when deriving the motion equations of a rigid body in matrix-vector form (and without the use of summation notation) using Lagrangian principles while using a constrained or unconstrained parameterization of SO(3). A particularly interesting and novel approach presented is to parameterize the attitude using the rotation matrix itself, explicitly considering the orthonormality constraint; doing so is exposed within an example. Attention is paid to the virtual work term in Lagrange's equation, where one of the identities proves very useful when deriving the generalized forces/torques. The development of the virtual work term in the derivation of the rigid-body equations has not received much attention in the literature.

The second novel contribution is the investigation of the effect of perturbations of parameterizations of SO(3). In particular, it is shown how perturbations in one parameterization of a rotation matrix impact different parameterizations of the same rotation matrix. The results can be used in sensitivity and error analysis, for example.

The third novel contribution of this paper lies in the specific examples presented. The identities related to the time-rate-of-change and partial derivatives of SO(3) parameterizations are used to derive the translational and rotational motion equations of a rigid body in matrix-vector form in a concise manner. Rotational equations of motion are considered as a special case. The perturbation results are used to assess how the potential energy of a rigid body is perturbed given small perturbations in attitude. Finally, one of the identities and the perturbation results are used in a culminating example where a rigorous treatment of the Extended Kalman Filter (EKF) for spacecraft attitude estimation using minimal attitude representations is presented. These three examples highlight the utility, and significance, of the our results in many fields of engineering.

The remainder of this paper is organized as follows. General identities for unconstrained parameterizations of SO(3) are derived in Section 2. Section 3 presents their counterparts for constrained parameterizations of SO(3). Section 4 presents general results on perturbations of parameterizations of SO(3). In Sections 5, 6 and 7, these identities and perturbation results are applied to the derivation of the dynamic equations describing rigid-body motion using Lagrangian principles, determining the sensitivity of a bodies potential energy due to perturbations in attitude, and issues related to spacecraft attitude determination. Finally, Section 8 contains concluding remarks.

2 Unconstrained Parameterizations of SO(3)

We shall first consider unconstrained parameterizations of SO(3) where

$$\text{SO(3)} = \left\{ \mathbf{C} \in \mathbb{R}^{3\times 3} \mid \mathbf{C}^T\mathbf{C} = \mathbf{1},\ \det[\mathbf{C}] = +1 \right\},$$

where $\mathbf{1}$ is the identity matrix. Consider a rotation matrix $\mathbf{C} \in$ SO(3), also called a direction cosine matrix [1]. Note that some authors refer to $\mathbf{C}^T$ as the rotation matrix [5]; we follow the convention of [1] where $\mathbf{C}$ is called the rotation matrix. Let $\mathbf{p} \in \mathbb{R}^n$ be any unconstrained parameterization of $\mathbf{C}$. Note that this includes independent three-set parameterizations ($n = 3$) such as Euler angles and Rodrigues parameters. That is, let

$$\mathbf{C} = \mathbf{C}(\mathbf{p}),$$

such that the mapping $\mathbf{C}(\mathbf{p}) : \mathbb{R}^n \to \text{SO(3)}$ is twice continuously differentiable. Such a situation might arise when $\mathbf{C}$ is the result of a sequence of rotational transformations, such as

$$\mathbf{C} = \mathbf{C}_m(\mathbf{p}_m)\mathbf{C}_{m-1}(\mathbf{p}_{m-1})\cdots\mathbf{C}_1(\mathbf{p}_1),$$

where $\mathbf{p}_i \in \mathbb{R}^{n_i}$, with $n_i \in \{1,2,3\}$, are unconstrained parameterizations of the rotation matrices $\mathbf{C}_i$, for $i = 1,\ldots,m$. Then, the combined vector $\mathbf{p}^T \triangleq \begin{bmatrix} \mathbf{p}_1^T & \cdots & \mathbf{p}_m^T \end{bmatrix}$ forms an n-dimensional unconstrained parameterization of $\mathbf{C}$, where $n = \sum_{i=1}^{m} n_i$. A special case of this is the rotation matrix resulting from an Euler rotation sequence.

The kinematics of the rotation matrix is given by Poisson's equation [1], which is

$$\dot{\mathbf{C}} = -\omega^{\times}\mathbf{C}, \tag{1}$$

where $\omega \in \mathbb{R}^3$ is the angular velocity associated with $\mathbf{C}$, and

$$\mathbf{a}^{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

is the skew-symmetric cross-product operator associated with the column matrix $\mathbf{a} = [a_1\ a_2\ a_3]^T$. Let us write the rotation matrix as

$$\mathbf{C}^T = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix}, \tag{2}$$

where $\mathbf{c}_i$ are the columns of $\mathbf{C}^T$ for $i = 1,2,3$. We immediately have

$$\mathbf{C}^T\mathbf{C} = \mathbf{c}_1\mathbf{c}_1^T + \mathbf{c}_2\mathbf{c}_2^T + \mathbf{c}_3\mathbf{c}_3^T = \mathbf{1}. \tag{3}$$

We also have

$$\mathbf{C}\mathbf{C}^T = \begin{bmatrix} \mathbf{c}_1^T\mathbf{c}_1 & \mathbf{c}_1^T\mathbf{c}_2 & \mathbf{c}_1^T\mathbf{c}_3 \\ \mathbf{c}_2^T\mathbf{c}_1 & \mathbf{c}_2^T\mathbf{c}_2 & \mathbf{c}_2^T\mathbf{c}_3 \\ \mathbf{c}_3^T\mathbf{c}_1 & \mathbf{c}_3^T\mathbf{c}_2 & \mathbf{c}_3^T\mathbf{c}_3 \end{bmatrix} = \mathbf{1}. \tag{4}$$

Since $\mathbf{p}$ is unconstrained, we may differentiate (4) with respect to the i^{th} component of $\mathbf{p}$, to obtain

$$\frac{\partial}{\partial p_i}\left[\mathbf{C}\mathbf{C}^T\right] = \begin{bmatrix} 2\mathbf{c}_1^T\frac{\partial \mathbf{c}_1}{\partial p_i} & \mathbf{c}_1^T\frac{\partial \mathbf{c}_2}{\partial p_i} + \mathbf{c}_2^T\frac{\partial \mathbf{c}_1}{\partial p_i} & \mathbf{c}_1^T\frac{\partial \mathbf{c}_3}{\partial p_i} + \mathbf{c}_3^T\frac{\partial \mathbf{c}_1}{\partial p_i} \\ \mathbf{c}_1^T\frac{\partial \mathbf{c}_2}{\partial p_i} + \mathbf{c}_2^T\frac{\partial \mathbf{c}_1}{\partial p_i} & 2\mathbf{c}_2^T\frac{\partial \mathbf{c}_2}{\partial p_i} & \mathbf{c}_2^T\frac{\partial \mathbf{c}_3}{\partial p_i} + \mathbf{c}_3^T\frac{\partial \mathbf{c}_2}{\partial p_i} \\ \mathbf{c}_1^T\frac{\partial \mathbf{c}_3}{\partial p_i} + \mathbf{c}_3^T\frac{\partial \mathbf{c}_1}{\partial p_i} & \mathbf{c}_2^T\frac{\partial \mathbf{c}_3}{\partial p_i} + \mathbf{c}_3^T\frac{\partial \mathbf{c}_2}{\partial p_i} & 2\mathbf{c}_3^T\frac{\partial \mathbf{c}_3}{\partial p_i} \end{bmatrix} = \mathbf{0}, \tag{5}$$

which leads to

$$\mathbf{c}_i^T \frac{\partial \mathbf{c}_j}{\partial \mathbf{p}^T} = \begin{cases} \mathbf{0}, & i = j, \\ -\mathbf{c}_j^T \partial \mathbf{c}_i / \partial \mathbf{p}^T, & i \neq j, \end{cases} \tag{6}$$

for $i, j = 1, 2, 3$, where we use the notation

$$\frac{\partial \mathbf{a}}{\partial \mathbf{p}^T} = \begin{bmatrix} \frac{\partial \mathbf{a}}{\partial p_1} & \cdots & \frac{\partial \mathbf{a}}{\partial p_n} \end{bmatrix}.$$

Next, let us obtain a kinematic relationship between $\dot{\mathbf{p}}$ and ω. Using (2), we can rearrange (1) to obtain

$$\begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} = -\omega^{\times} = \dot{\mathbf{C}}\mathbf{C}^T = \begin{bmatrix} \dot{\mathbf{c}}_1^T \mathbf{c}_1 & \dot{\mathbf{c}}_1^T \mathbf{c}_2 & \dot{\mathbf{c}}_1^T \mathbf{c}_3 \\ \dot{\mathbf{c}}_2^T \mathbf{c}_1 & \dot{\mathbf{c}}_2^T \mathbf{c}_2 & \dot{\mathbf{c}}_2^T \mathbf{c}_3 \\ \dot{\mathbf{c}}_3^T \mathbf{c}_1 & \dot{\mathbf{c}}_3^T \mathbf{c}_2 & \dot{\mathbf{c}}_3^T \mathbf{c}_3 \end{bmatrix},$$

from which we obtain

$$\dot{\mathbf{c}}_i^T \mathbf{c}_j = \begin{cases} 0, & i = j, \\ -\dot{\mathbf{c}}_j^T \mathbf{c}_i, & i \neq j, \end{cases} \tag{7}$$

for $i, j = 1, 2, 3$, and likewise, we extract the angular velocity

$$\omega = \begin{bmatrix} \dot{\mathbf{c}}_2^T \mathbf{c}_3 \\ \dot{\mathbf{c}}_3^T \mathbf{c}_1 \\ \dot{\mathbf{c}}_1^T \mathbf{c}_2 \end{bmatrix}. \tag{8}$$

Applying the chain-rule, we have

$$\dot{\mathbf{c}}_i = \frac{\partial \mathbf{c}_i}{\partial \mathbf{p}^T} \dot{\mathbf{p}}, \quad i = 1, 2, 3, \tag{9}$$

and substituting (9) into (8), we obtain the following expression for the angular velocity

$$\omega = \mathbf{S}(\mathbf{p})\dot{\mathbf{p}}, \tag{10}$$

where

$$\mathbf{S}(\mathbf{p}) = \begin{bmatrix} \mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} \\ \mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} \\ \mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} \end{bmatrix}. \tag{11}$$

Now, let $\bar{\mathbf{S}}(\mathbf{p})$ be any other continuously differentiable matrix satisfying

$$\omega = \bar{\mathbf{S}}(\mathbf{p})\dot{\mathbf{p}}. \tag{12}$$

Since $\mathbf{p}$ is unconstrained, so is $\dot{\mathbf{p}}$, and since (10) and (12) must hold for all $\dot{\mathbf{p}}$, it follows that the matrix $\bar{\mathbf{S}}(\mathbf{p}) = \mathbf{S}(\mathbf{p})$, and the set of all $\bar{\mathbf{S}}(\mathbf{p})$ satisfying (12) is uniquely given by $\mathbf{S}(\mathbf{p})$ satisfying (11). We shall now derive three identities for unconstrained parameterizations.

2.1 Identity 1

We will now derive an expression for

$$\frac{\partial(\mathbf{C}\mathbf{v})}{\partial \mathbf{p}^T},$$

where $\mathbf{v} \in \mathbb{R}^3$ is not a function of $\mathbf{p}$. We note that

$$\mathbf{C}\mathbf{v} = \begin{bmatrix} \mathbf{c}_1^T\mathbf{v} \\ \mathbf{c}_2^T\mathbf{v} \\ \mathbf{c}_3^T\mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{v}^T\mathbf{c}_1 \\ \mathbf{v}^T\mathbf{c}_2 \\ \mathbf{v}^T\mathbf{c}_3 \end{bmatrix}. \tag{13}$$

Hence, we have

$$\frac{\partial(\mathbf{C}\mathbf{v})}{\partial\mathbf{p}^T} = \begin{bmatrix} \mathbf{v}^T\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} \end{bmatrix}. \tag{14}$$

Using (11) and (13), let us now form the product

$$(\mathbf{C}\mathbf{v})^\times\mathbf{S}(\mathbf{p}) = \begin{bmatrix} 0 & -\mathbf{v}^T\mathbf{c}_3 & \mathbf{v}^T\mathbf{c}_2 \\ \mathbf{v}^T\mathbf{c}_3 & 0 & -\mathbf{v}^T\mathbf{c}_1 \\ -\mathbf{v}^T\mathbf{c}_2 & \mathbf{v}^T\mathbf{c}_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_3^T\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} \\ \mathbf{c}_1^T\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} \\ \mathbf{c}_2^T\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T} \end{bmatrix} = \begin{bmatrix} \mathbf{v}^T\left(-\mathbf{c}_3\mathbf{c}_1^T\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} + \mathbf{c}_2\mathbf{c}_2^T\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\right) \\ \mathbf{v}^T\left(\mathbf{c}_3\mathbf{c}_3^T\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} - \mathbf{c}_1\mathbf{c}_2^T\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\right) \\ \mathbf{v}^T\left(-\mathbf{c}_2\mathbf{c}_3^T\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} + \mathbf{c}_1\mathbf{c}_1^T\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\right) \end{bmatrix}. \tag{15}$$

Making use of the identities in (6), this becomes

$$(\mathbf{C}\mathbf{v})^\times\mathbf{S}(\mathbf{p}) = \begin{bmatrix} \mathbf{v}^T\left(\mathbf{c}_3\mathbf{c}_3^T + \mathbf{c}_2\mathbf{c}_2^T\right)\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\left(\mathbf{c}_3\mathbf{c}_3^T + \mathbf{c}_1\mathbf{c}_1^T\right)\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\left(\mathbf{c}_2\mathbf{c}_2^T + \mathbf{c}_1\mathbf{c}_1^T\right)\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} \end{bmatrix}.$$

Finally, noting from (6) that $\mathbf{c}_i^T\partial\mathbf{c}_i/\partial\mathbf{p}^T = \mathbf{0}$ for $i = 1,2,3$, and making use of (3), we obtain

$$(\mathbf{C}\mathbf{v})^\times\mathbf{S}(\mathbf{p}) = \begin{bmatrix} \mathbf{v}^T\left(\mathbf{c}_3\mathbf{c}_3^T + \mathbf{c}_2\mathbf{c}_2^T + \mathbf{c}_1\mathbf{c}_1^T\right)\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\left(\mathbf{c}_3\mathbf{c}_3^T + \mathbf{c}_1\mathbf{c}_1^T + \mathbf{c}_2\mathbf{c}_2^T\right)\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\left(\mathbf{c}_2\mathbf{c}_2^T + \mathbf{c}_1\mathbf{c}_1^T + \mathbf{c}_3\mathbf{c}_3^T\right)\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} \end{bmatrix} = \begin{bmatrix} \mathbf{v}^T\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} \\ \mathbf{v}^T\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} \end{bmatrix}. \tag{16}$$

Comparing (16) with (14), we obtain the identity

$$\frac{\partial(\mathbf{C}\mathbf{v})}{\partial\mathbf{p}^T} = (\mathbf{C}\mathbf{v})^\times\mathbf{S}(\mathbf{p}). \tag{17}$$

This identity has been derived in [6] directly for Euler angle parameterizations of SO(3). We have just demonstrated universality of this identity for any unconstrained parameterization of SO(3).

2.2 Identity 2

An expression for

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T}$$

will now be derived, where $\mathbf{v} \in \mathbb{R}^3$ is not a function of $\mathbf{p}$. First, by definition,

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T} = \begin{bmatrix} \frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial p_1} & \cdots & \frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial p_n} \end{bmatrix}.$$

Now, for $i = 1, \ldots, n$, we have

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial p_i} = \frac{\partial(\mathbf{C}^T)}{\partial p_i}\mathbf{v}.$$

Since $\mathbf{C}^T = \mathbf{C}^{-1}$, we have [3]

$$\frac{\partial(\mathbf{C}^T)}{\partial p_i} = -\mathbf{C}^T\frac{\partial(\mathbf{C})}{\partial p_i}\mathbf{C}^T. \tag{18}$$

Therefore, it follows that

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial p_i} = -\mathbf{C}^T\frac{\partial(\mathbf{C})}{\partial p_i}\mathbf{C}^T\mathbf{v} = -\mathbf{C}^T\frac{\partial(\mathbf{C}\overline{\mathbf{v}})}{\partial p_i},$$

where $\overline{\mathbf{v}} = \mathbf{C}^T\mathbf{v}$ is understood to be held fixed in the differentiation. Consequently, we have

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T} = -\mathbf{C}^T\frac{\partial(\mathbf{C}\overline{\mathbf{v}})}{\partial\mathbf{p}^T}. \tag{19}$$

Making use of Identity 1 in (17), we have

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T} = -\mathbf{C}^T\left(\mathbf{C}\overline{\mathbf{v}}\right)^\times\mathbf{S}(\mathbf{p}).$$

Substituting for $\overline{\mathbf{v}}$, we obtain the second identity

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T} = -\mathbf{C}^T\mathbf{v}^\times\mathbf{S}(\mathbf{p}). \tag{20}$$

Although this identity is presented in [7] when the attitude is parameterized using Euler angles, we have just shown that it holds for any unconstrained parameterization of SO(3).

2.3 Identity 3

We shall now find an identity for $\dot{\mathbf{S}}(\mathbf{p})$. Differentiating (11) with respect to time, we obtain

$$\dot{\mathbf{S}}(\mathbf{p}) = \mathbf{A} + \mathbf{B}, \tag{21}$$

where

$$\mathbf{A} = \begin{bmatrix} \dot{\mathbf{c}}_3^T \frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} \\ \dot{\mathbf{c}}_1^T \frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} \\ \dot{\mathbf{c}}_2^T \frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \mathbf{c}_3^T \frac{d}{dt}\left[\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T}\right] \\ \mathbf{c}_1^T \frac{d}{dt}\left[\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\right] \\ \mathbf{c}_2^T \frac{d}{dt}\left[\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\right] \end{bmatrix}. \tag{22}$$

Making use of (9), $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} \dot{\mathbf{p}}^T \left(\frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\right)^T \frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T} \\ \dot{\mathbf{p}}^T \left(\frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\right)^T \frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T} \\ \dot{\mathbf{p}}^T \left(\frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T}\right)^T \frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T} \end{bmatrix}, \tag{23}$$

Let us now develop $\mathbf{B}$. First, we note that for $i = 1,2,3$,

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial\mathbf{c}_i}{\partial\mathbf{p}^T}\right] &= \frac{d}{dt}\left[\frac{\partial\mathbf{c}_i}{\partial p_1} \cdots \frac{\partial\mathbf{c}_i}{\partial p_n}\right], \\ &= \left[\frac{\partial}{\partial\mathbf{p}^T}\left(\frac{\partial\mathbf{c}_i}{\partial p_1}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial\mathbf{p}^T}\left(\frac{\partial\mathbf{c}_i}{\partial p_n}\right)\dot{\mathbf{p}}\right], \\ &= \left[\frac{\partial}{\partial p_1}\left(\frac{\partial\mathbf{c}_i}{\partial\mathbf{p}^T}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial p_n}\left(\frac{\partial\mathbf{c}_i}{\partial\mathbf{p}^T}\right)\dot{\mathbf{p}}\right]. \end{aligned}$$

Substituting this into (22), we obtain

$$\mathbf{B} = \begin{bmatrix} \mathbf{c}_3^T \begin{bmatrix} \frac{\partial}{\partial p_1}\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial p_n}\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \end{bmatrix} \\ \mathbf{c}_1^T \begin{bmatrix} \frac{\partial}{\partial p_1}\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial p_n}\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \end{bmatrix} \\ \mathbf{c}_2^T \begin{bmatrix} \frac{\partial}{\partial p_1}\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial p_n}\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \end{bmatrix} \end{bmatrix}. \tag{24}$$

Let us now form $\partial\omega/\partial\mathbf{p}^T$. Using (10), we have for $i = 1, \ldots, n$

$$\frac{\partial \omega}{\partial p_i} = \begin{bmatrix} \left(\frac{\partial \mathbf{c}_3}{\partial p_i}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\dot{\mathbf{p}} + \mathbf{c}_3^T \frac{\partial}{\partial p_i}\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \\ \left(\frac{\partial \mathbf{c}_1}{\partial p_i}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\dot{\mathbf{p}} + \mathbf{c}_1^T \frac{\partial}{\partial p_i}\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \\ \left(\frac{\partial \mathbf{c}_2}{\partial p_i}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\dot{\mathbf{p}} + \mathbf{c}_2^T \frac{\partial}{\partial p_i}\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \end{bmatrix}, \tag{25}$$

from which we obtain

$$\frac{\partial \omega}{\partial \mathbf{p}^T} = \mathbf{D} + \mathbf{B}, \tag{26}$$

where $\mathbf{B}$ is given in (24), and

$$\mathbf{D} = \begin{bmatrix} \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} \\ \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} \\ \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} \end{bmatrix}. \tag{27}$$

Let us now evaluate $\omega^\times \mathbf{S}(\mathbf{p})$. From (10) and (11), we have

$$\begin{aligned} \omega^\times \mathbf{S}(\mathbf{p}) &= \begin{bmatrix} 0 & -\dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2 & \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1 \\ \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2 & 0 & -\dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3 \\ -\dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1 & \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} \\ \mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} \\ \mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} \end{bmatrix}, \\ &= \begin{bmatrix} \dot{\mathbf{p}}^T \left(-\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2 \mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} + \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1 \mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right) \\ \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2 \mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} - \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3 \mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right) \\ \dot{\mathbf{p}}^T \left(-\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1 \mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} + \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3 \mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right) \end{bmatrix}. \end{aligned} \tag{28}$$

Making use of (6), we can rewrite this at

$$\omega^{\times}\mathbf{S}(\mathbf{p}) = \begin{bmatrix} \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1\mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} - \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1\mathbf{c}_1^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} \right) \\ \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2\mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} - \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2\mathbf{c}_2^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} \right) \\ \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3\mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} - \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3\mathbf{c}_3^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} \right) \end{bmatrix}.$$

Noting from (6) that $\mathbf{c}_i^T \partial \mathbf{c}_i / \partial \mathbf{p}^T = \mathbf{0}$ for $i = 1,2,3$, and making use of (3), we obtain

$$\omega^{\times}\mathbf{S}(\mathbf{p}) = \begin{bmatrix} \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} - \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} \right) \\ \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} - \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} \right) \\ \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} - \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} \right) \end{bmatrix} \quad (29)$$

Comparing (29) to (23) and (27), we find that

$$\mathbf{D} = \omega^{\times}\mathbf{S}(\mathbf{p}) + \mathbf{A}, \quad (30)$$

such that from (26), we have

$$\frac{\partial \omega}{\partial \mathbf{p}^T} = \omega^{\times}\mathbf{S}(\mathbf{p}) + \mathbf{A} + \mathbf{B}. \quad (31)$$

Making use of (21), and rearranging we have the final identity

$$\dot{\mathbf{S}}(\mathbf{p}) = \frac{\partial \omega}{\partial \mathbf{p}^T} - \omega^{\times}\mathbf{S}(\mathbf{p}). \quad (32)$$

This identity has been presented in [8] for Euler angle parameterizations of SO(3). We have just demonstrated universality of this identity for any unconstrained attitude parameterization of attitude. If $\mathbf{S}(\mathbf{p})$ is invertible (and consequently $\mathbf{p} \in \mathbb{R}^3$), (32) can be rearranged into the form

$$\left(\dot{\mathbf{S}}(\mathbf{p}) - \frac{\partial \omega}{\partial \mathbf{p}^T}\right)\mathbf{S}^{-1}(\mathbf{p}) = -\omega^{\times}. \quad (33)$$

These are the well-known Hamel coefficients used in the derivation of Euler's equation for rigid-body motion using Lagrangian principles [9, 10], as will be seen in Section 4. The expression in (33) has been presented in [11] and [12] for Euler angle parameterizations of SO(3), and they were proven to hold for all unconstrained parameterizations with $\mathbf{p} \in \mathbb{R}^3$ in [13, 14]. The derivation presented here is distinctly different from that of [13, 14]. In particular, in [13, 14] summation notation is heavily used.

3 Constrained Parameterizations

We shall now find analogous identities for constrained parameterizations of SO(3). Again, consider a rotation matrix $\mathbf{C} \in \mathrm{SO}(3)$, and now let $\mathbf{p} \in \mathbb{R}^n$ be any constrained parameterization of $\mathbf{C}$, with $n > 3$. Since SO(3) is a 3-dimensional manifold, we consider parameterizations which satisfy the constraint

$$\Phi(\mathbf{p}) = \mathbf{0}, \tag{34}$$

for some $\Phi(\mathbf{p}) : \mathbb{R}^n \to \mathbb{R}^{n-3}$, and we define the set

$$\mathcal{P} = \{\mathbf{p} \in \mathbb{R}^n : \Phi(\mathbf{p}) = \mathbf{0}\}. \tag{35}$$

It is assumed that $\Phi(\mathbf{p})$ is twice continuously differentiable on $\mathcal{P}$. Consider parameterizations such that the mapping $\mathbf{C}(\mathbf{p}) : \mathcal{P} \to \mathrm{SO}(3)$ is twice continuously differentiable.

Associated with the constraint (34) is the differential constraint

$$\Xi(\mathbf{p})\dot{\mathbf{p}} = \mathbf{0}, \tag{36}$$

where

$$\Xi(\mathbf{p}) = \frac{\partial \Phi}{\partial \mathbf{p}^T}. \tag{37}$$

It is assumed that $\Xi(\mathbf{p})$ has full row rank for all $\mathbf{p} \in \mathcal{P}$, which ensures that $\mathcal{P}$ is a 3-dimensional manifold. It also ensures that for any $\mathbf{p}^* \in \mathcal{P}$, and any $\mathbf{y} \in \mathcal{N}(\Xi(\mathbf{p}^*))$ (where $\mathcal{N}(\Xi(\mathbf{p}))$ denotes the null-space of $\Xi(\mathbf{p})$), there exists a differentiable curve $\mathbf{p}(t) \in \mathcal{P}$ with $t \in [a,b]$ for some $a, b \in \mathbb{R}$ such that $\mathbf{p}(t^*) = \mathbf{p}^*$ and $\dot{\mathbf{p}}(t^*) = \mathbf{y}$ for some $t^* \in [a,b]$ [15]. That is, the tangent space of $\mathcal{P}$ at any point $\mathbf{p} \in \mathcal{P}$ coincides with the null-space of $\Xi(\mathbf{p})$. Note that these assumptions are satisfied by constrained

4-set parameterizations ($n = 4$) such as quaternions and axis-angle parameters, or even the rotation matrix itself ($n = 9$), in which cases the mappings are in fact surjective. For example, in the case of the quaternion, $\Phi(\mathbf{p}) = \mathbf{p}^T\mathbf{p} - 1$, and $\mathcal{P}$ is the unit sphere in $\mathbb{R}^4$.

It is readily verified that the developments leading to equations (10) and (11) for unconstrained parameterizations are valid for constrained parameterizations also. However, for (10) to hold, we must have $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ must satisfy the constraint (36). Unlike the unconstrained case, the set of all continuously differentiable (on $\mathcal{P}$) $\bar{\mathbf{S}}(\mathbf{p})$ satisfying (12) for all $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ satisfying (36), is not unique. We shall deal with this non-uniqueness in Section 3.4. In the following we shall proceed with $\mathbf{S}(\mathbf{p})$ as defined in (11).

Since $\mathbf{p}$ must satisfy (34), equations (5) and (6) do not hold for constrained parameterizations, since $\mathbf{p}$ cannot be arbitrarily varied. Hence, we seek analogous relationships for constrained $\mathbf{p}$.

Equations (7) and (9) are valid for constrained parameterizations provided $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ satisfies (36). Therefore, substituting (9) into (7), we see that the relations

$$\mathbf{c}_i^T \frac{\partial \mathbf{c}_j}{\partial \mathbf{p}^T}\dot{\mathbf{p}} = \begin{cases} \mathbf{0}, & i = j, \\ -\mathbf{c}_j^T(\partial \mathbf{c}_i/\partial \mathbf{p}^T)\dot{\mathbf{p}}, & i \neq j, \end{cases} \tag{38}$$

must hold for all $\mathbf{p} \in \mathcal{P}$ and all $\dot{\mathbf{p}}$ satisfying (36), for $i, j = 1, 2, 3$.

We now combine (10) and (36) as

$$\begin{bmatrix} \omega \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{S}(\mathbf{p}) \\ \Xi(\mathbf{p}) \end{bmatrix} \dot{\mathbf{p}}. \tag{39}$$

Assuming invertibility of (39) for some $\mathbf{p} \in \mathcal{P}$, we denote

$$\begin{bmatrix} \Gamma(\mathbf{p}) \ \alpha(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} \mathbf{S}(\mathbf{p}) \\ \Xi(\mathbf{p}) \end{bmatrix}^{-1}, \tag{40}$$

as the inverse at $\mathbf{p} \in \mathcal{P}$, where $\Gamma(\mathbf{p}) \in \mathbb{R}^{n\times 3}$ and $\alpha(\mathbf{p}) \in \mathbb{R}^{n\times(n-3)}$.

Then, inverting (39), we obtain

$$\dot{\mathbf{p}} = \Gamma(\mathbf{p})\omega. \tag{41}$$

Furthermore, from (40), we have

$$\begin{bmatrix} \mathbf{S}(\mathbf{p}) \\ \Xi(\mathbf{p}) \end{bmatrix} \begin{bmatrix} \Gamma(\mathbf{p}) & \alpha(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix},$$

from which we obtain

$$\mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) = \mathbf{1},\ \Xi(\mathbf{p})\Gamma(\mathbf{p}) = \mathbf{0}, \tag{42}$$

for all $\mathbf{p} \in \mathcal{P}$ where the inverse in (40) exists. Substituting (41) into (38), we find that

$$\mathbf{c}_i^T \frac{\partial \mathbf{c}_j}{\partial \mathbf{p}^T}\Gamma(\mathbf{p})\omega = \begin{cases} \mathbf{0}, & i = j, \\ -\mathbf{c}_j^T(\partial \mathbf{c}_i/\partial \mathbf{p}^T)\Gamma(\mathbf{p})\omega, & i \neq j, \end{cases}$$

Since these must hold for all $\omega \in \mathbb{R}^3$, we have

$$\mathbf{c}_i^T \frac{\partial \mathbf{c}_j}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) = \begin{cases} \mathbf{0}, & i = j, \\ -\mathbf{c}_j^T(\partial \mathbf{c}_i/\partial \mathbf{p}^T)\Gamma(\mathbf{p}), & i \neq j, \end{cases} \tag{43}$$

for all $\mathbf{p} \in \mathcal{P}$ where the inverse in (40) exists, and all $i, j = 1,2,3$.

We now proceed with the derivation of identities analogous to those in Sections 2.1 to 2.3.

3.1 Identity 1′

Analogous to Identity 1, we shall now seek an expression for

$$\frac{\partial(\mathbf{C}\mathbf{v})}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}),$$

where $\mathbf{v} \in \mathbb{R}^3$ is independent of $\mathbf{p}$, and $\mathbf{p} \in \mathcal{P}$ such that the inverse in (40) exists. From (13), we obtain

$$\frac{\partial(\mathbf{C}\mathbf{v})}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) = \begin{bmatrix} \mathbf{v}^T \frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \\ \mathbf{v}^T \frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \\ \mathbf{v}^T \frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \end{bmatrix}. \tag{44}$$

Using (11) and (13), in a similar manner to (15), let us now form the product

$$(\mathbf{C}\mathbf{v})^\times \mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) = \begin{bmatrix} \mathbf{v}^T \left(-\mathbf{c}_3\mathbf{c}_1^T \frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) + \mathbf{c}_2\mathbf{c}_2^T \frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\Gamma(\mathbf{p})\right) \\ \mathbf{v}^T \left(\mathbf{c}_3\mathbf{c}_3^T \frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) - \mathbf{c}_1\mathbf{c}_2^T \frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\Gamma(\mathbf{p})\right) \\ \mathbf{v}^T \left(-\mathbf{c}_2\mathbf{c}_3^T \frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) + \mathbf{c}_1\mathbf{c}_1^T \frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\Gamma(\mathbf{p})\right) \end{bmatrix}.$$

Making use of the identities in (43), this becomes

$$(\mathbf{C}\mathbf{v})^\times \mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) = \begin{bmatrix} \mathbf{v}^T \left(\mathbf{c}_3\mathbf{c}_3^T + \mathbf{c}_2\mathbf{c}_2^T\right) \frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \\ \mathbf{v}^T \left(\mathbf{c}_3\mathbf{c}_3^T + \mathbf{c}_1\mathbf{c}_1^T\right) \frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \\ \mathbf{v}^T \left(\mathbf{c}_2\mathbf{c}_2^T + \mathbf{c}_1\mathbf{c}_1^T\right) \frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \end{bmatrix}.$$

Finally, noting from (43) that $\mathbf{c}_i^T(\partial\mathbf{c}_i/\partial\mathbf{p}^T)\Gamma(\mathbf{p}) = \mathbf{0}$ for $i = 1,2,3$, in a similar manner to (16), we obtain

$$(\mathbf{C}\mathbf{v})^\times \mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) = \begin{bmatrix} \mathbf{v}^T \frac{\partial\mathbf{c}_1}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \\ \mathbf{v}^T \frac{\partial\mathbf{c}_2}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \\ \mathbf{v}^T \frac{\partial\mathbf{c}_3}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) \end{bmatrix}. \tag{45}$$

Comparing (45) with (44), and making use of (42) we obtain the identity analogous to (17)

$$\frac{\partial(\mathbf{C}\mathbf{v})}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) = (\mathbf{C}\mathbf{v})^\times. \tag{46}$$

for all $\mathbf{p} \in \mathcal{P}$ such that the inverse in (40) exists.

3.2 Identity 2′

Analogous to Identity 2, we now find an expression for

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}),$$

where $\mathbf{v} \in \mathbb{R}^3$ is independent of $\mathbf{p}$ and $\mathbf{p} \in \mathcal{P}$ such that the inverse in (40) exists.

In a similar manner to the development leading to (19), we find that

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) = -\mathbf{C}^T\frac{\partial(\mathbf{C}\bar{\mathbf{v}})}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}), \tag{47}$$

where $\bar{\mathbf{v}} = \mathbf{C}^T\mathbf{v}$ is understood to be held fixed in the differentiation. Making use of Identity 1 in (46), we have

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) = -\mathbf{C}^T\left(\mathbf{C}\bar{\mathbf{v}}\right)^{\times}.$$

Substituting for $\bar{\mathbf{v}}$, we obtain the identity analogous to (20)

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial\mathbf{p}^T}\Gamma(\mathbf{p}) = -\mathbf{C}^T\mathbf{v}^{\times}, \tag{48}$$

for all $\mathbf{p} \in \mathcal{P}$ such that the inverse in (40) exists.

3.3 Identity 3′

Analogous to Identity 3, we shall now find an identity for $\dot{\mathbf{S}}(\mathbf{p})\Gamma(\mathbf{p})$ where $\mathbf{p} \in \mathcal{P}$ such that the inverse in (40) exists. Upon examination of the development from equation (21) to (27), it is readily apparent that it is also valid for constrained parameterizations provided $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ satisfies (36). Therefore, post-multiplying equations (21) to (27) by $\Gamma(\mathbf{p})$, we obtain

$$\dot{\mathbf{S}}(\mathbf{p})\Gamma(\mathbf{p}) = \bar{\mathbf{A}} + \bar{\mathbf{B}}, \tag{49}$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} \Gamma(\mathbf{p}) \\ \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} \Gamma(\mathbf{p}) \\ \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} \Gamma(\mathbf{p}) \end{bmatrix}, \tag{50}$$

and

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{c}_3^T \left[\frac{\partial}{\partial p_1}\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial p_n}\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \right] \Gamma(\mathbf{p}) \\ \mathbf{c}_1^T \left[\frac{\partial}{\partial p_1}\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial p_n}\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \right] \Gamma(\mathbf{p}) \\ \mathbf{c}_2^T \left[\frac{\partial}{\partial p_1}\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \cdots \frac{\partial}{\partial p_n}\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)\dot{\mathbf{p}} \right] \Gamma(\mathbf{p}) \end{bmatrix}. \tag{51}$$

Likewise,

$$\frac{\partial \omega}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) = \bar{\mathbf{D}} + \bar{\mathbf{B}}, \tag{52}$$

where $\bar{\mathbf{B}}$ is given in (51), and

$$\bar{\mathbf{D}} = \begin{bmatrix} \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T} \Gamma(\mathbf{p}) \\ \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T} \Gamma(\mathbf{p}) \\ \dot{\mathbf{p}}^T \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T} \Gamma(\mathbf{p}) \end{bmatrix}. \tag{53}$$

Similar to (28), we obtain

$$\omega^\times \mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) = \begin{bmatrix} \dot{\mathbf{p}}^T \left(-\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2 \mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) + \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1 \mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) \right) \\ \dot{\mathbf{p}}^T \left(\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_2 \mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) - \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3 \mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) \right) \\ \dot{\mathbf{p}}^T \left(-\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_1 \mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) + \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \mathbf{c}_3 \mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) \right) \end{bmatrix}.$$

Similar to the development leading to (29), we obtain using (38), (43) and (3) that

$$\omega^{\times}\mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) = \begin{bmatrix} \dot{\mathbf{p}}^T\left(\left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) - \left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\Gamma(\mathbf{p})\right) \\ \dot{\mathbf{p}}^T\left(\left(\frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) - \left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\Gamma(\mathbf{p})\right) \\ \dot{\mathbf{p}}^T\left(\left(\frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) - \left(\frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\right)^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\Gamma(\mathbf{p})\right) \end{bmatrix} \quad (54)$$

Comparing (54) to (50) and (53), we find that

$$\bar{\mathbf{D}} = \omega^{\times}\mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) + \bar{\mathbf{A}}, \quad (55)$$

such that from (52), we have

$$\frac{\partial \omega}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) = \omega^{\times}\mathbf{S}(\mathbf{p})\Gamma(\mathbf{p}) + \bar{\mathbf{A}} + \bar{\mathbf{B}}. \quad (56)$$

Making use of (49) and (42), and rearranging we have the identity analogous to (32)

$$\left(\dot{\mathbf{S}}(\mathbf{p}) - \frac{\partial \omega}{\partial \mathbf{p}^T}\right)\Gamma(\mathbf{p}) = -\omega^{\times}, \quad (57)$$

for all $\mathbf{p} \in \mathcal{P}$ such that the inverse in (40) exists and all $\dot{\mathbf{p}}$ satisfying (36).

3.4 Non-Uniqueness of $\bar{\mathbf{S}}(\mathbf{p})$

As mentioned previously, the set of all continuously differentiable $\bar{\mathbf{S}}(\mathbf{p})$ satisfying (12) for all $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ satisfying (36), is not unique. In addition, the matrix $\Xi(\mathbf{p})$ in (36) defining the tangent space of $\mathcal{P}$ at point $\mathbf{p} \in \mathcal{P}$ can also be replaced by a whole family of continuously differentiable matrices $\bar{\Xi}(\mathbf{p})$ yielding the same tangent space, as will be seen shortly. In this subsection it is shown that the identities obtained in Sections 3.1 to 3.3 are independent of the choice of $\bar{\mathbf{S}}(\mathbf{p})$ and $\bar{\Xi}(\mathbf{p})$ in (12) and (36), respectively.

Consider the matrix

$$\bar{\mathbf{S}}(\mathbf{p}) = \mathbf{S}(\mathbf{p}) + \mathbf{F}(\mathbf{p})\Xi(\mathbf{p}), \quad (58)$$

where $\mathbf{F}(\mathbf{p}) \in \mathbb{R}^{3\times(n-3)}$ is continuously differentiable on $\mathcal{P}$. Then, using (10) and (36), it is clear that $\bar{\mathbf{S}}(\mathbf{p})$ is continuously differentiable on $\mathcal{P}$ and satisfies (12) for all $\dot{\mathbf{p}}$ satisfying (36). Conversely, consider a continuously differentiable matrix $\bar{\mathbf{S}}(\mathbf{p}) \in \mathbb{R}^{3\times n}$ on $\mathcal{P}$, satisfying (12) for all $\dot{\mathbf{p}}$ satisfying (36). Then, equating (10) and (12) it follows that

$$\left(\bar{\mathbf{S}}(\mathbf{p}) - \mathbf{S}(\mathbf{p})\right)\dot{\mathbf{p}} = \mathbf{0}, \tag{59}$$

for all $\dot{\mathbf{p}}$ satisfying (36). This means that the columns of $\bar{\mathbf{S}}^T(\mathbf{p}) - \mathbf{S}^T(\mathbf{p})$ lie in the orthogonal complement of $\mathcal{N}(\Xi(\mathbf{p}))$. Now, the orthogonal complement of $\mathcal{N}(\Xi(\mathbf{p}))$ is given by the range space of $\Xi^T(\mathbf{p})$ [16]. As such, each column of $\bar{\mathbf{S}}^T(\mathbf{p}) - \mathbf{S}^T(\mathbf{p})$ can be written as a linear combination of the columns of $\Xi^T(\mathbf{p})$, and therefore there exists a matrix $\mathbf{F}(\mathbf{p}) \in \mathbb{R}^{3\times(n-3)}$ such that

$$\bar{\mathbf{S}}^T(\mathbf{p}) - \mathbf{S}^T(\mathbf{p}) = \Xi^T(\mathbf{p})\mathbf{F}^T(\mathbf{p}). \tag{60}$$

Since $\Xi(\mathbf{p})$ has full row-rank (by assumption), this can be solved for $\mathbf{F}^T(\mathbf{p})$ as

$$\mathbf{F}^T(\mathbf{p}) = \left(\Xi(\mathbf{p})\Xi^T(\mathbf{p})\right)^{-1}\Xi(\mathbf{p})\left(\bar{\mathbf{S}}^T(\mathbf{p}) - \mathbf{S}^T(\mathbf{p})\right). \tag{61}$$

By continuous differentiability (on $\mathcal{P}$) of $\bar{\mathbf{S}}(\mathbf{p})$, $\mathbf{S}(\mathbf{p})$, $\Xi(\mathbf{p})$ and the matrix inverse, it follows that $\mathbf{F}(\mathbf{p})$ is continuously differentiable on $\mathcal{P}$. Therefore, (60) can now be rearranged into the form of (58), and we conclude that all continuously differentiable $\bar{\mathbf{S}}(\mathbf{p})$ satisfying (12) for all $\dot{\mathbf{p}}$ satisfying (36), can be written in the form of (58) for some continuously differentiable $\mathbf{F}(\mathbf{p}) \in \mathbb{R}^{3\times(n-3)}$ on $\mathcal{P}$.

Next, we obtain a full characterization of the constraint (36), which itself is non-unique. To this end, consider the matrix

$$\bar{\Xi}(\mathbf{p}) = \mathbf{G}(\mathbf{p})\Xi(\mathbf{p}). \tag{62}$$

where $\mathbf{G}(\mathbf{p}) \in \mathbb{R}^{(n-3)\times(n-3)}$ is continuously differentiable and invertible on $\mathcal{P}$. Then, the constraint in (36) may be equivalently written as

$$\bar{\Xi}(\mathbf{p})\dot{\mathbf{p}} = \mathbf{0}. \tag{63}$$

Conversely, consider any $\bar{\Xi}(\mathbf{p}) \in \mathbb{R}^{(n-3)\times n}$, which is continuously differentiable on $\mathcal{P}$ such that $\mathcal{N}(\bar{\Xi}(\mathbf{p})) = \mathcal{N}(\Xi(\mathbf{p}))$ for all $\mathbf{p} \in \mathcal{P}$ (note that this implies that $\bar{\Xi}(\mathbf{p})$ has full row-rank). It is readily seen that the constraint in (63) with this choice of $\bar{\Xi}(\mathbf{p})$ is equivalent to (36). Then, each row in $\bar{\Xi}(\mathbf{p})$ can be written as a linear combination of the rows in $\Xi(\mathbf{p})$. In a similar manner as was used to find $\mathbf{F}(\mathbf{p})$ in (61), we can find a $\mathbf{G}(\mathbf{p})$ such that $\bar{\Xi}(\mathbf{p})$ in the form of (62). This $\mathbf{G}(\mathbf{p})$ is given by

$$\mathbf{G}(\mathbf{p}) = \bar{\Xi}(\mathbf{p})\Xi^T(\mathbf{p})\left(\Xi(\mathbf{p})\Xi^T(\mathbf{p})\right)^{-1}, \tag{64}$$

which is continuously differentiable on $\mathcal{P}$. From (64), it follows that to show invertibility $\mathbf{G}(\mathbf{p})$, it suffices to show that $\Xi(\mathbf{p})\bar{\Xi}(\mathbf{p})^T\mathbf{x} \neq \mathbf{0}$ for any $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^{n-3}$. To this end, since $\bar{\Xi}(\mathbf{p})$ has full row-rank, it follows that $\mathbf{y} = \bar{\Xi}(\mathbf{p})^T\mathbf{x} \neq \mathbf{0}$ whenever $\mathbf{x} \neq \mathbf{0}$. Next, $\mathbf{y}$ lies in the space spanned by the columns of $\bar{\Xi}(\mathbf{p})^T$, which is perpendicular to $\mathcal{N}(\bar{\Xi}(\mathbf{p}))$. Since $\mathcal{N}(\bar{\Xi}(\mathbf{p})) = \mathcal{N}(\Xi(\mathbf{p}))$, it follows that $\mathbf{y}$ is perpendicular to $\mathcal{N}(\Xi(\mathbf{p}))$ also, and therefore $\Xi(\mathbf{p})\mathbf{y} = \Xi(\mathbf{p})\bar{\Xi}(\mathbf{p})^T\mathbf{x} \neq \mathbf{0}$ whenever $\mathbf{x} \neq \mathbf{0}$. This shows that $\mathbf{G}(\mathbf{p})$ is invertible. In conclusion, all continuously differentiable (on $\mathcal{P}$) $\bar{\Xi}(\mathbf{p}) \in \mathbb{R}^{(n-3)\times n}$ for which (63) is equivalent to the constraint in (36) can be written in the form of (62) where $\mathbf{G}(\mathbf{p}) \in \mathbb{R}^{(n-3)\times(n-3)}$ is continuously differentiable and invertible on $\mathcal{P}$.

Now, assembling (12) and (63) in the same manner as (39), and using (58) and (62), we have

$$\begin{bmatrix} \omega \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{S}}(\mathbf{p}) \\ \bar{\Xi}(\mathbf{p}) \end{bmatrix} \dot{\mathbf{p}} = \begin{bmatrix} \mathbf{1} & \mathbf{F}(\mathbf{p}) \\ \mathbf{0} & \mathbf{G}(\mathbf{p}) \end{bmatrix} \begin{bmatrix} \mathbf{S}(\mathbf{p}) \\ \Xi(\mathbf{p}) \end{bmatrix} \dot{\mathbf{p}}. \tag{65}$$

Now, clearly since $\mathbf{G}(\mathbf{p})$ is invertible, equation (65) is invertible if and only if (39) is invertible. Furthermore, utilizing (40), we have that when it is invertible for $\mathbf{p} \in \mathcal{P}$,

$$\begin{bmatrix} \bar{\mathbf{S}}(\mathbf{p}) \\ \bar{\Xi}(\mathbf{p}) \end{bmatrix}^{-1} = \begin{bmatrix} \Gamma(\mathbf{p}) & -\Gamma(\mathbf{p})\mathbf{F}(\mathbf{p})\mathbf{G}(\mathbf{p})^{-1} + \alpha(\mathbf{p})\mathbf{G}(\mathbf{p})^{-1} \end{bmatrix}, \tag{66}$$

resulting again in

$$\bar{\mathbf{S}}(\mathbf{p})\Gamma(\mathbf{p}) = \mathbf{1},\ \bar{\Xi}(\mathbf{p})\Gamma(\mathbf{p}) = \mathbf{0}, \tag{67}$$

for all $\mathbf{p} \in \mathcal{P}$ where the inverse in (66) (equivalently (40)) exists. In addition, we recover (41) once more. In particular, this shows that $\Gamma(\mathbf{p})$ is the unique matrix satisfying (41) and (67).

Since $\Gamma(\mathbf{p})$ is unique, it doesn't matter how it is obtained (it does not necessarily have to come from the inversion in (66)), as long as it satisfies (41), and identities $1'$ and $2'$ in (46) and (48) automatically hold.

Next, we would like to verify that identity $3'$ in (57) holds for any choice of $\bar{\mathbf{S}}(\mathbf{p})$ used to represent the kinematics in (12). To this end, we evaluate

$$\left(\frac{d}{dt}\left(\bar{\mathbf{S}}(\mathbf{p})\right) - \frac{\partial\left(\bar{\mathbf{S}}(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T}\right)\Gamma(\mathbf{p}), \tag{68}$$

where $\bar{\mathbf{S}}(\mathbf{p}) \in \mathbb{R}^{3\times n}$ is continuously differentiable. Using the characterization in (58), this becomes

$$\begin{aligned}\left(\frac{d}{dt}\left(\bar{\mathbf{S}}(\mathbf{p})\right) - \frac{\partial\left(\bar{\mathbf{S}}(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T}\right)\Gamma(\mathbf{p}) &= \left(\dot{\mathbf{S}}(\mathbf{p}) - \frac{\partial\left(\mathbf{S}(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T}\right)\Gamma(\mathbf{p}) + \left(\frac{d}{dt}\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\right) - \frac{\partial\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T}\right)\Gamma(\mathbf{p}),\\ &= -\omega^\times + \left(\frac{d}{dt}\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\right) - \frac{\partial\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T}\right)\Gamma(\mathbf{p}),\end{aligned} \tag{69}$$

where (57) has been used on the first term on the right hand side. We shall now deal with the final term in (69). First, we note that

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\right)\Gamma(\mathbf{p}) &= \frac{d}{dt}\left(\mathbf{F}(\mathbf{p})\right)\Xi(\mathbf{p})\Gamma(\mathbf{p}) + \mathbf{F}(\mathbf{p})\frac{d}{dt}\left(\Xi(\mathbf{p})\right)\Gamma(\mathbf{p}),\\ &= \mathbf{F}(\mathbf{p})\frac{d}{dt}\left(\Xi(\mathbf{p})\right)\Gamma(\mathbf{p}),\end{aligned} \tag{70}$$

where (42) has been used to eliminate the first term on the right hand side. Next,

$$\begin{aligned}\frac{\partial\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T} &= \left[\tfrac{\partial\mathbf{F}(\mathbf{p})}{\partial p_1}\Xi(\mathbf{p})\dot{\mathbf{p}} \cdots \tfrac{\partial\mathbf{F}(\mathbf{p})}{\partial p_n}\Xi(\mathbf{p})\dot{\mathbf{p}}\right] + \mathbf{F}(\mathbf{p})\frac{\partial\left(\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T},\\ &= \mathbf{F}(\mathbf{p})\frac{\partial\left(\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T},\end{aligned} \tag{71}$$

where (36) has been used to eliminate the first term on the right hand side. Consequently, using (70) and (71), the residual term in (69) becomes

$$\left(\frac{d}{dt}\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\right) - \frac{\partial\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T}\right)\Gamma(\mathbf{p}) = \mathbf{F}(\mathbf{p})\left(\frac{d}{dt}\left(\Xi(\mathbf{p})\right) - \frac{\partial\left(\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial\mathbf{p}^T}\right)\Gamma(\mathbf{p}). \tag{72}$$

Let us now partition the constraint function in (34) as

$$\Phi(\mathbf{p}) = \begin{bmatrix} \phi_1(\mathbf{p}) \\ \vdots \\ \phi_{n-3}(\mathbf{p}) \end{bmatrix}, \tag{73}$$

where $\phi_i(\mathbf{p}) : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, \ldots, n-3$. Then, the matrix $\Xi(\mathbf{p})$ in (37) is given by

$$\Xi(\mathbf{p}) = \begin{bmatrix} \frac{\partial \phi_1}{\partial \mathbf{p}^T} \\ \vdots \\ \frac{\partial \phi_{n-3}}{\partial \mathbf{p}^T} \end{bmatrix}. \tag{74}$$

Consequently, its time-derivative is given by

$$\frac{d}{dt}\left(\Xi(\mathbf{p})\right) = \begin{bmatrix} \dot{\mathbf{p}}^T \frac{\partial^2 \phi_1}{\partial \mathbf{p} \partial \mathbf{p}^T} \\ \vdots \\ \dot{\mathbf{p}}^T \frac{\partial^2 \phi_{n-3}}{\partial \mathbf{p} \partial \mathbf{p}^T} \end{bmatrix}, \tag{75}$$

where $\partial^2 \phi_i / \partial \mathbf{p} \partial \mathbf{p}^T$ is the Hessian of ϕ_i for $i = 1, \ldots, n-3$. By continuous differentiability of $\Phi(\mathbf{p})$, it is symmetric. Next, using (74), we have

$$\frac{\partial \left(\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial \mathbf{p}^T} = \begin{bmatrix} \frac{\partial}{\partial \mathbf{p}^T}\left(\frac{\partial \phi_1}{\partial \mathbf{p}^T}\dot{\mathbf{p}}\right) \\ \vdots \\ \frac{\partial}{\partial \mathbf{p}^T}\left(\frac{\partial \phi_{n-3}}{\partial \mathbf{p}^T}\dot{\mathbf{p}}\right) \end{bmatrix}. \tag{76}$$

Now,

$$\begin{aligned} \frac{\partial}{\partial \mathbf{p}^T}\left(\frac{\partial \phi_i}{\partial \mathbf{p}^T}\dot{\mathbf{p}}\right) &= \frac{\partial}{\partial \mathbf{p}^T}\left(\dot{\mathbf{p}}^T \frac{\partial \phi_i}{\partial \mathbf{p}}\right), \\ &= \dot{\mathbf{p}}^T \frac{\partial}{\partial \mathbf{p}^T}\left(\frac{\partial \phi_i}{\partial \mathbf{p}}\right), \\ &= \dot{\mathbf{p}}^T \frac{\partial^2 \phi_i}{\partial \mathbf{p} \partial \mathbf{p}^T}. \end{aligned} \tag{77}$$

Substitution of (77) into (76) and comparing to (75) shows that

$$\frac{d}{dt}\left(\Xi(\mathbf{p})\right) = \frac{\partial\left(\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial \mathbf{p}^T},$$

and consequently (72) becomes

$$\left(\frac{d}{dt}\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\right) - \frac{\partial\left(\mathbf{F}(\mathbf{p})\Xi(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial \mathbf{p}^T}\right)\Gamma(\mathbf{p}) = \mathbf{0},$$

and finally (69) becomes

$$\left(\dot{\bar{\mathbf{S}}}(\mathbf{p}) - \frac{\partial\left(\bar{\mathbf{S}}(\mathbf{p})\dot{\mathbf{p}}\right)}{\partial \mathbf{p}^T}\right)\Gamma(\mathbf{p}) = -\omega^\times, \tag{78}$$

which is the desired result. In conclusion, we have shown that identity $3'$ in (57) holds for any choice of $\bar{\mathbf{S}}(\mathbf{p})$ which is continuously differentiable on $\mathcal{P}$ and satisfies (12) for all $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ satisfying (63). The importance of this result is that in any practical application of identity $3'$, it does not matter how the matrix $\bar{\mathbf{S}}(\mathbf{p})$ is found (it is not necessary to evaluate (11)), as long as it is continuously differentiable on $\mathcal{P}$ and satisfies (12) for all $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ satisfying (63).

3.5 Example - Parameterizing the Rotation matrix by Itself

Suppose that we parameterize the rotation matrix by itself, by setting

$$\mathbf{p} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}. \tag{79}$$

Since $\mathbf{C}^T\mathbf{C} = \mathbf{1}$ and $\det[\mathbf{C}] = +1$, we define the constraint function in (34) to be

$$\Phi(\mathbf{p}) = \begin{bmatrix} \mathbf{c}_1^T\mathbf{c}_1 - 1 \\ \mathbf{c}_2^T\mathbf{c}_2 - 1 \\ \mathbf{c}_1^T\mathbf{c}_2 \\ \mathbf{c}_1^\times\mathbf{c}_2 - \mathbf{c}_3 \end{bmatrix}. \tag{80}$$

Accordingly, the differential constraint in (36) has

$$\Xi(\mathbf{p}) = \begin{bmatrix} \mathbf{c}_1^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_2^T & \mathbf{0} \\ \mathbf{c}_2^T & \mathbf{c}_1^T & \mathbf{0} \\ -\mathbf{c}_2^\times & \mathbf{c}_1^\times & -\mathbf{1} \end{bmatrix}. \tag{81}$$

It is straightforward to show that the kinematics in (10) has

$$\mathbf{S}(\mathbf{p}) = \begin{bmatrix} \mathbf{0} & \mathbf{c}_3^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{c}_1^T \\ \mathbf{c}_2^T & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{82}$$

The inverse kinematics in (41) can also readily be found as follows. Transposing Poisson's equations, we have

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{c}}_1 & \dot{\mathbf{c}}_2 & \dot{\mathbf{c}}_3 \end{bmatrix} = \dot{\mathbf{C}}^T &= \mathbf{C}^T \omega^\times \\ &= \mathbf{C}^T \omega^\times \mathbf{C}\mathbf{C}^T \\ &= \left(\mathbf{C}^T \omega\right)^\times \mathbf{C}^T \\ &= \left(\mathbf{C}^T \omega\right)^\times \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix}. \end{aligned}$$

Note that we have made use of the identity $\left(\mathbf{C}^T\mathbf{a}\right)^\times = \mathbf{C}^T\mathbf{a}^\times\mathbf{C}$ [1]. Writing this componentwise, we have for $i = 1,2,3$

$$\begin{aligned} \dot{\mathbf{c}}_i &= \left(\mathbf{C}^T \omega\right)^\times \mathbf{c}_i \\ &= -\mathbf{c}_i^\times \mathbf{C}^T \omega \\ &= -\mathbf{c}_i^\times \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} \omega, \end{aligned}$$

which leads to

$$\begin{bmatrix} \dot{\mathbf{c}}_1 \\ \dot{\mathbf{c}}_2 \\ \dot{\mathbf{c}}_3 \end{bmatrix} = \begin{bmatrix} -\mathbf{c}_1^\times \mathbf{c}_1 & -\mathbf{c}_1^\times \mathbf{c}_2 & -\mathbf{c}_1^\times \mathbf{c}_3 \\ -\mathbf{c}_2^\times \mathbf{c}_1 & -\mathbf{c}_2^\times \mathbf{c}_2 & -\mathbf{c}_2^\times \mathbf{c}_3 \\ -\mathbf{c}_3^\times \mathbf{c}_1 & -\mathbf{c}_3^\times \mathbf{c}_2 & -\mathbf{c}_3^\times \mathbf{c}_3 \end{bmatrix} \omega.$$

Since $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ form a right-handed triad, we have

$$\mathbf{c}_1^{\times}\mathbf{c}_2 = \mathbf{c}_3,\ \mathbf{c}_2^{\times}\mathbf{c}_3 = \mathbf{c}_1,\ \mathbf{c}_3^{\times}\mathbf{c}_1 = \mathbf{c}_2.$$

Therefore, we have

$$\begin{bmatrix} \dot{\mathbf{c}}_1 \\ \dot{\mathbf{c}}_2 \\ \dot{\mathbf{c}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{c}_3 & \mathbf{c}_2 \\ \mathbf{c}_3 & \mathbf{0} & -\mathbf{c}_1 \\ -\mathbf{c}_2 & \mathbf{c}_1 & \mathbf{0} \end{bmatrix} \omega,$$

from which we identify

$$\Gamma(\mathbf{p}) = \begin{bmatrix} \mathbf{0} & -\mathbf{c}_3 & \mathbf{c}_2 \\ \mathbf{c}_3 & \mathbf{0} & -\mathbf{c}_1 \\ -\mathbf{c}_2 & \mathbf{c}_1 & \mathbf{0} \end{bmatrix}. \tag{83}$$

It is now straightforward to verify by direct multiplication that (42) hold for all $\mathbf{p}$ satisfying (34) with (80).

We now verify identities $1'$, $2'$ and $3'$. First, we verify identity $1'$. We have

$$\frac{\partial(\mathbf{Cv})}{\partial \mathbf{p}^T} = \begin{bmatrix} \mathbf{v}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}^T \end{bmatrix},$$

such that

$$\begin{aligned} \frac{\partial(\mathbf{Cv})}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) &= \begin{bmatrix} \mathbf{v}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{c}_3 & \mathbf{c}_2 \\ \mathbf{c}_3 & \mathbf{0} & -\mathbf{c}_1 \\ -\mathbf{c}_2 & \mathbf{c}_1 & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & -\mathbf{v}^T\mathbf{c}_3 & \mathbf{v}^T\mathbf{c}_2 \\ \mathbf{v}^T\mathbf{c}_3 & \mathbf{0} & -\mathbf{v}^T\mathbf{c}_1 \\ -\mathbf{v}^T\mathbf{c}_2 & \mathbf{v}^T\mathbf{c}_1 & \mathbf{0} \end{bmatrix} \\ &= (\mathbf{Cv})^{\times}, \end{aligned}$$

which is precisely identity $1'$ in (46).

For identity $2'$, we write

$$\mathbf{C}^T\mathbf{v} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} \mathbf{v},$$

from which we obtain

$$\frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial \mathbf{p}^T} = \begin{bmatrix} v_1\mathbf{1} & v_2\mathbf{1} & v_3\mathbf{1} \end{bmatrix},$$

where $\mathbf{v} = [v_1, v_2, v_3]^T$. Therefore, we have

$$\begin{aligned} \frac{\partial(\mathbf{C}^T\mathbf{v})}{\partial \mathbf{p}^T}\Gamma(\mathbf{p}) &= \begin{bmatrix} v_1\mathbf{1} & v_2\mathbf{1} & v_3\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{c}_3 & \mathbf{c}_2 \\ \mathbf{c}_3 & \mathbf{0} & -\mathbf{c}_1 \\ -\mathbf{c}_2 & \mathbf{c}_1 & \mathbf{0} \end{bmatrix} \\ &= -\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \\ &= -\mathbf{C}^T\mathbf{v}^\times, \end{aligned}$$

which is precisely identity $2'$ in (48).

Finally, for identity $3'$, we note from (8) that

$$\frac{\partial \omega}{\partial \mathbf{p}^T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dot{\mathbf{c}}_2^T \\ \dot{\mathbf{c}}_3^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{c}}_1^T & \mathbf{0} \end{bmatrix}.$$

Likewise, differentiating (82), we obtain

$$\dot{\mathbf{S}}(\mathbf{p}) = \begin{bmatrix} \mathbf{0} & \dot{\mathbf{c}}_3^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dot{\mathbf{c}}_1^T \\ \dot{\mathbf{c}}_2^T & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Therefore, we have

$$
\begin{aligned}
\left(\dot{\mathbf{S}}(\mathbf{p}) - \frac{\partial \omega}{\partial \mathbf{p}^T}\right)\Gamma(\mathbf{p}) &= \begin{bmatrix} \mathbf{0} & \dot{\mathbf{c}}_3^T & -\dot{\mathbf{c}}_2^T \\ -\dot{\mathbf{c}}_3^T & \mathbf{0} & \dot{\mathbf{c}}_1^T \\ \dot{\mathbf{c}}_2^T & -\dot{\mathbf{c}}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{c}_3 & \mathbf{c}_2 \\ \mathbf{c}_3 & \mathbf{0} & -\mathbf{c}_1 \\ -\mathbf{c}_2 & \mathbf{c}_1 & \mathbf{0} \end{bmatrix}, \\
&= \begin{bmatrix} \dot{\mathbf{c}}_3^T \mathbf{c}_3 + \dot{\mathbf{c}}_2^T \mathbf{c}_2 & -\dot{\mathbf{c}}_2^T \mathbf{c}_1 & -\dot{\mathbf{c}}_3^T \mathbf{c}_1 \\ -\dot{\mathbf{c}}_1^T \mathbf{c}_2 & \dot{\mathbf{c}}_3^T \mathbf{c}_3 + \dot{\mathbf{c}}_1^T \mathbf{c}_1 & -\dot{\mathbf{c}}_3^T \mathbf{c}_2 \\ -\dot{\mathbf{c}}_1^T \mathbf{c}_3 & -\dot{\mathbf{c}}_2^T \mathbf{c}_3 & \dot{\mathbf{c}}_2^T \mathbf{c}_2 + \dot{\mathbf{c}}_1^T \mathbf{c}_1 \end{bmatrix}.
\end{aligned}
$$

Making use of relationships (7) and (8), this becomes

$$
\left(\dot{\mathbf{S}}(\mathbf{p}) - \frac{\partial \omega}{\partial \mathbf{p}^T}\right)\Gamma(\mathbf{p}) = -\omega^{\times},
$$

which is precisely identity $3'$ in (57).

Finally, we note that we have demonstrated identities $1'$ to $3'$ hold globally, for all $\mathbf{p}$ and $\dot{\mathbf{p}}$, satisfying (34) with (80) and (36) with (81), respectively.

4 Perturbations on SO(3)

Let $\mathbf{p}$ be any parameterization (either unconstrained or constrained) of the rotation matrix $\mathbf{C}(\mathbf{p})$. We seek the effect of a perturbation $\delta\mathbf{p}$ on $\mathbf{C}(\mathbf{p})$. That is, we seek an approximate expression for $\mathbf{C}(\mathbf{p}+\delta\mathbf{p})$. Before proceeding, we take the first variation of (4), to obtain the relationships

$$
\mathbf{c}_i^T \frac{\partial \mathbf{c}_j}{\partial \mathbf{p}^T}\delta\mathbf{p} = \begin{cases} 0, & i = j, \\ -\mathbf{c}_j^T \frac{\partial \mathbf{c}_i}{\partial \mathbf{p}^T}\delta\mathbf{p}, & i \neq j, \end{cases} \tag{84}
$$

for $i, j = 1,2,3$. Now, we note from (6) that these relationships are trivially satisfied for unconstrained parameterizations, and that any variation $\delta\mathbf{p} \in \mathbb{R}^n$ is allowed in this case. However, for constrained parameterizations, we have the variational constraint obtained by taking the first variation of (34)

$$
\Xi(\mathbf{p})\delta\mathbf{p} = \mathbf{0}, \tag{85}
$$

where $\Xi(\mathbf{p})$ is defined in (37). Note that (86) is equivalently written as

$$\bar{\Xi}(\mathbf{p})\delta\mathbf{p} = \mathbf{0}, \tag{86}$$

where $\bar{\Xi}(\mathbf{p})$ is given in (62). Therefore, for constrained parameterizations, only variations satisfying (86) are allowed.

Now, let us define the rotation matrix

$$\mathbf{C}_{err} \triangleq \mathbf{C}(\mathbf{p}+\delta\mathbf{p})\mathbf{C}^T(\mathbf{p}). \tag{87}$$

Equivalently, $\mathbf{C}(\mathbf{p}+\delta\mathbf{p}) = \mathbf{C}_{err}\mathbf{C}(\mathbf{p})$, such that $\mathbf{C}_{err}$ represents the rotational transformation from the original frame (represented by $\mathbf{C}(\mathbf{p})$) to the perturbed frame (represented by $\mathbf{C}(\mathbf{p}+\delta\mathbf{p})$). We shall seek a first order expression for $\mathbf{C}_{err}$ in $\delta\mathbf{p}$. It will prove easier to work with $\mathbf{C}_{err}^T$.

$$\mathbf{C}_{err}^T = \mathbf{C}(\mathbf{p})\mathbf{C}^T(\mathbf{p}+\delta\mathbf{p}). \tag{88}$$

Now, using (2), we obtain a first order Taylor series expansion for (88) as

$$\begin{aligned}
\mathbf{C}_{err}^T &\approx \mathbf{C}(\mathbf{p})\left(\mathbf{C}^T(\mathbf{p}) + \begin{bmatrix} \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\delta\mathbf{p} & \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\delta\mathbf{p} & \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\delta\mathbf{p} \end{bmatrix}\right), \\
&= \mathbf{1} + \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \mathbf{c}_3^T \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\delta\mathbf{p} & \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\delta\mathbf{p} & \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\delta\mathbf{p} \end{bmatrix}.
\end{aligned}$$

Making use of (84), this becomes

$$\mathbf{C}_{err}^T \approx \mathbf{1} + \begin{bmatrix} 0 & -\mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\delta\mathbf{p} & \mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\delta\mathbf{p} \\ \mathbf{c}_2^T \frac{\partial \mathbf{c}_1}{\partial \mathbf{p}^T}\delta\mathbf{p} & 0 & -\mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\delta\mathbf{p} \\ -\mathbf{c}_1^T \frac{\partial \mathbf{c}_3}{\partial \mathbf{p}^T}\delta\mathbf{p} & \mathbf{c}_3^T \frac{\partial \mathbf{c}_2}{\partial \mathbf{p}^T}\delta\mathbf{p} & 0 \end{bmatrix}.$$

Now, from (11), we see that this is just

$$\mathbf{C}_{err}^{T} \approx \mathbf{1} + (\mathbf{S}(\mathbf{p})\delta\mathbf{p})^{\times},$$

and consequently we have to first order

$$\mathbf{C}_{err} = \mathbf{C}(\mathbf{p}+\delta\mathbf{p})\mathbf{C}^{T}(\mathbf{p}) \approx \mathbf{1} - (\mathbf{S}(\mathbf{p})\delta\mathbf{p})^{\times}, \tag{89}$$

or equivalently to first order,

$$\mathbf{C}(\mathbf{p}+\delta\mathbf{p}) \approx \left(\mathbf{1} - (\mathbf{S}(\mathbf{p})\delta\mathbf{p})^{\times}\right)\mathbf{C}(\mathbf{p}). \tag{90}$$

By the analysis from section 3.4, for constrained parameterizations (90) are equivalent to

$$\mathbf{C}_{err} = \mathbf{C}(\mathbf{p}+\delta\mathbf{p})\mathbf{C}^{T}(\mathbf{p}) \approx \mathbf{1} - \left(\bar{\mathbf{S}}(\mathbf{p})\delta\mathbf{p}\right)^{\times}, \tag{91}$$

and

$$\mathbf{C}(\mathbf{p}+\delta\mathbf{p}) \approx \left(\mathbf{1} - \left(\bar{\mathbf{S}}(\mathbf{p})\delta\mathbf{p}\right)^{\times}\right)\mathbf{C}(\mathbf{p}), \tag{92}$$

for any $\bar{\mathbf{S}}(\mathbf{p})$ which is continuously differentiable on $\mathcal{P}$ and satisfies (12) for all $\mathbf{p} \in \mathcal{P}$ and $\dot{\mathbf{p}}$ satisfying (63).

This result appears in [6], but not in the general form presented here where $\mathbf{p}$ could be any rotation matrix parameterization. Let us now parameterize $\mathbf{C}_{err}$ by the rotation vector $\boldsymbol{\Phi} = \mathbf{a}\phi$, where $\mathbf{a}$ and ϕ are Euler's principal axis and angle of rotation, respectively. Correspondingly, we have [1]

$$\mathbf{C}_{err}(\boldsymbol{\Phi}) = \mathbf{1} + \frac{(1-\cos\phi)}{\phi^2}\boldsymbol{\Phi}^{\times}\boldsymbol{\Phi}^{\times} - \frac{\sin\phi}{\phi}\boldsymbol{\Phi}^{\times}, \tag{93}$$

and associated kinematic matrix

$$\mathbf{S}(\Phi) = \mathbf{1} - \frac{(1-\cos\phi)}{\phi^2}\Phi^{\times} + \frac{(\phi-\sin\phi)}{\phi^3}\Phi^{\times}\Phi^{\times}, \tag{94}$$

where $\phi = \|\Phi\|$. We note that as written, both $\mathbf{C}_{err}(\Phi)$ and $\mathbf{S}(\Phi)$ are undefined at $\Phi = \mathbf{0}$. Instead, let us make use of the series expansions

$$\frac{\sin\phi}{\phi} = \sum_{n=0}^{\infty} \frac{(-1)^n(\Phi^T\Phi)^n}{(2n+1)!} =: f_1(\Phi),$$

$$\frac{1-\cos\phi}{\phi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n(\Phi^T\Phi)^n}{(2n+2)!} =: f_2(\Phi),$$

$$\frac{\phi-\sin\phi}{\phi^3} = \sum_{n=0}^{\infty} \frac{(-1)^n(\Phi^T\Phi)^n}{(2n+3)!} =: f_3(\Phi).$$

It is readily shown that $f_1(\Phi)$, $f_2(\Phi)$ and $f_3(\Phi)$ are well-defined, and are continuously differentiable on $\mathbb{R}^3$. Hence, we rewrite (93) and (94) as

$$\mathbf{C}_{err}(\Phi) = \mathbf{1} + f_2(\Phi)\Phi^{\times}\Phi^{\times} - f_1(\Phi)\Phi^{\times}, \tag{95}$$

and

$$\mathbf{S}(\Phi) = \mathbf{1} - f_2(\Phi)\Phi^{\times} + f_3(\Phi)\Phi^{\times}\Phi^{\times}, \tag{96}$$

which are continuously differentiable on $\mathbb{R}^3$.

Clearly, from (87) when $\delta\mathbf{p} = \mathbf{0}$, we have $\mathbf{C}_{err}(\Phi) = \mathbf{1}$ and consequently $\Phi = \mathbf{0}$. Therefore, by continuity, it follows that for small enough $\delta\mathbf{p}$, the rotation vector ϕ can also be considered to be small. Let us accordingly write $\Phi = \delta\Phi$. Now, we have from (90) and (96) (with $\mathbf{p}$ replaced by Φ) that to first order in $\delta\Phi$ (with $\Phi = \mathbf{0}$),

$$\mathbf{C}_{err}(\delta\Phi) \approx \mathbf{1} - \delta\Phi^{\times}. \tag{97}$$

Comparing (91) and (97), we have

$$\delta\Phi = \bar{\mathbf{S}}(\mathbf{p})\delta\mathbf{p}. \tag{98}$$

Note that for unconstrained parameterizations, $\bar{\mathbf{S}}(\mathbf{p})$ is simply $\mathbf{S}(\mathbf{p})$ by uniqueness. If $\mathbf{p} \in \mathbb{R}^3$, and $\mathbf{S}(\mathbf{p})$ is invertible, the equation (98) can be inverted to obtain

$$\delta\mathbf{p} = \mathbf{S}^{-1}(\mathbf{p})\delta\Phi. \tag{99}$$

If $\mathbf{p}$ is constrained to $\mathbf{p} \in \mathcal{P}$, we append (86) to (98) to obtain

$$\begin{bmatrix} \delta\Phi \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{S}}(\mathbf{p}) \\ \bar{\Xi}(\mathbf{p}) \end{bmatrix} \delta\mathbf{p}. \tag{100}$$

Using (66), we then obtain

$$\delta\mathbf{p} = \Gamma(\mathbf{p})\delta\Phi, \tag{101}$$

for all $\mathbf{p} \in \mathcal{P}$ where the inverse in (100) exists, and where $\Gamma(\mathbf{p})$ satisfies equation (41).

5 Application: Boltzmann-Hamel Equations for a Rigid Body

We will now show how the obtained identities may be used to obtain the equations of motion for a rigid body.

5.1 Boltzmann-Hamel Equations

First, we provide a quick overview of Lagrange's equation, and their Boltzmann-Hamel variant [17, 18]. Consider a mechanical system whose configuration is fully described by the generalized coordinates $\mathbf{q} \in \mathbb{R}^m$. Suppose also that the system is subjected to the holonomic constraints

$$\pi(\mathbf{q}) = \mathbf{0}, \tag{102}$$

where $\pi : \mathbb{R}^m \to \mathbb{R}^p$ is continuously differentiable, and $1 < p < m$. We define the constraint set $\mathcal{Q} = \{\mathbf{q} : \pi(\mathbf{q}) = \mathbf{0}\}$.

The associated Lagrange equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q} + \Pi(\mathbf{q})^T \lambda, \tag{103}$$

where $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the Lagrangian, and $\mathbf{Q}$ is the generalized force vector, which is related to the virtual work $\delta W = \delta \mathbf{q}^T \mathbf{Q}$ for a virtual displacement $\delta \mathbf{q}$. The matrix

$$\Pi(\mathbf{q}) = \frac{\partial \pi}{\partial \mathbf{q}^T}, \tag{104}$$

is the Jacobian of the constraint (102). Finally, $\lambda \in \mathbb{R}^{m-p}$ is a Lagrange multiplier, such that $\Pi(\mathbf{q})^T \lambda$ is the generalized constraint force required to enforce (102).

Suppose now that we can find a generalized velocity

$$\mathbf{v} = \alpha(\mathbf{q})\dot{\mathbf{q}} \tag{105}$$

with $\mathbf{v} \in \mathbb{R}^\ell$ ($\ell = m - p$), such that the relationship

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \alpha(\mathbf{q}) \\ \Pi(\mathbf{q}) \end{bmatrix} \dot{\mathbf{q}},$$

is invertible for some $\mathbf{q} \in \mathcal{Q}$. Let the inverse relationship be given by

$$\dot{\mathbf{q}} = \begin{bmatrix} \beta(\mathbf{q}) & \eta(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix},$$

where $\beta : \mathbb{R}^\ell \to \mathbb{R}^m$ and $\eta : \mathbb{R}^p \to \mathbb{R}^m$. Then, we have

$$\dot{\mathbf{q}} = \beta(\mathbf{q})\mathbf{v}, \tag{106}$$

and

$$\alpha(\mathbf{q})\beta(\mathbf{q}) = \mathbf{1},\ \Pi(\mathbf{q})\beta(\mathbf{q}) = \mathbf{0}. \tag{107}$$

Suppose now that the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t) = \bar{L}(\mathbf{q}, \mathbf{v}, t)$ for all $\mathbf{q} \in \mathcal{Q}$. Then, we have

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = \left(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}^T}\right)^T \frac{\partial \bar{L}}{\partial \mathbf{v}},\ \frac{\partial L}{\partial \mathbf{q}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{q}^T}\right)^T \frac{\partial \bar{L}}{\partial \mathbf{v}} + \frac{\partial \bar{L}}{\partial \mathbf{q}}.$$

for all $\mathbf{q} \in \mathcal{Q}$. From (105), we have

$$\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}^T} = \alpha(\mathbf{q}),$$

so Lagrange's equation (103) becomes

$$\alpha(\mathbf{q})^T \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \mathbf{v}}\right) + \left(\dot{\alpha}(\mathbf{q}) - \frac{\partial \mathbf{v}}{\partial \mathbf{q}^T}\right)^T \frac{\partial \bar{L}}{\partial \mathbf{v}} - \frac{\partial \bar{L}}{\partial \mathbf{q}} = \mathbf{Q} + \Pi(\mathbf{q})^T \lambda. \tag{108}$$

Pre-multiplying (108) by $\beta(\mathbf{q})^T$, and using identities (107), we obtain the constrained Boltzmann-Hamel equations

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \mathbf{v}}\right) + \beta(\mathbf{q})^T \left(\dot{\alpha}(\mathbf{q}) - \frac{\partial \mathbf{v}}{\partial \mathbf{q}^T}\right)^T \frac{\partial \bar{L}}{\partial \mathbf{v}} - \beta(\mathbf{q})^T \frac{\partial \bar{L}}{\partial \mathbf{q}} = \beta(\mathbf{q})^T \mathbf{Q}. \tag{109}$$

Notice that the generalized constraint force has been eliminated. Finally, the Boltzmann-Hamel equations of motion must be augmented by the kinematical relationship given in (106), which ensures that $\mathbf{q}(t) \in \mathcal{Q}$.

Note that the term $\beta(\mathbf{q})^T \left(\dot{\alpha}(\mathbf{q}) - \left(\partial \mathbf{v}/\partial \mathbf{q}^T\right)\right)^T$ is termed the Hamel coefficient [10].

5.2 Rigid Body Dynamics

We shall now demonstrate how the presented identities can be used to derive the equations of motion for a rigid body from the Boltzmann-Hamel equations. We shall use vectrix notation developed by Hughes [1] for the initial formulation. Consider the rigid body as shown in Figure 1, where $\mathcal{F}_I$ is an inertial frame of reference, $\mathcal{F}_b$ is a body-fixed frame of reference with origin at point o, $\vec{\mathbf{r}}_o$ is the inertial position of o, $\vec{\rho}$ is the position of mass element dm from o and $\vec{\mathbf{f}}$ is the force per unit volume acting on dm. Finally, $\vec{\mathbf{r}}_{dm}$ is the inertial position of dm. Denoting an inertial time-derivative (time derivative relative

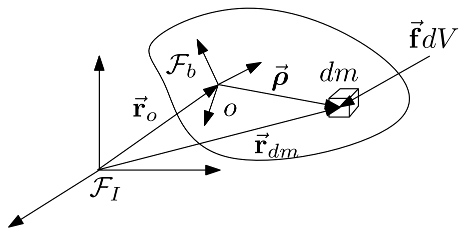

Fig. 1. Free-body diagram of a rigid body

to $\mathcal{F}_I$) by an overdot, the inertial velocity of dm is given by

$$\dot{\vec{\mathbf{r}}}_{dm} = \vec{\mathbf{v}}_o + \vec{\omega}\times\vec{\rho}, \tag{110}$$

where $\vec{\mathbf{v}}_o = \dot{\vec{\mathbf{r}}}_o$ is the inertial velocity of point o, and $\vec{\omega}$ is the inertial angular velocity of the rigid body.

Let us now express all vector quantities in terms of their coordinates in either $\mathcal{F}_I$ or $\mathcal{F}_b$, as

$$\vec{\mathbf{r}}_o = \vec{\mathcal{F}}_I^T\mathbf{r}_{o,I}, \;\vec{\omega} = \vec{\mathcal{F}}_b^T\omega, \;\vec{\rho} = \vec{\mathcal{F}}_b^T\rho, \;\vec{\mathbf{v}}_o = \vec{\mathcal{F}}_b^T\mathbf{v}_o,$$

and

$$\vec{\mathbf{f}} = \vec{\mathcal{F}}_b^T\mathbf{f}.$$

Note that all quantities are expressed in body coordinates except for the inertial position of o ($\vec{\mathbf{r}}_o$). Let us denote the rotation matrix transforming coordinates from $\mathcal{F}_I$ to $\mathcal{F}_b$ by $\mathbf{C}(\mathbf{p})$, which we parameterize by a constrained parameterization $\mathbf{p}\in\mathbb{R}^n$ as in Section 3. Then, we have

$$\mathbf{v}_o = \mathbf{C}(\mathbf{p})\dot{\mathbf{r}}_{o,I}, \tag{111}$$

for all $\mathbf{p} \in \mathcal{P}$. In addition, $\mathbf{C}(\mathbf{p})$ and ω satisfy the kinematic relationship given in (1).

In inertial coordinates, the position of element dm is given by

$$\vec{\mathbf{r}}_{dm} = \vec{\mathcal{F}}_I^T \left[\mathbf{r}_{o,I} + \mathbf{C}^T(\mathbf{p})\rho\right]. \tag{112}$$

Note that we require the position of dm in inertial coordinates, since to correctly determine the virtual work performed by external forces, the corresponding virtual displacement must be inertial [19].

From (112), it can be seen that the location of any mass element dm (with ρ given) is fully specified by $\mathbf{r}_{o,I}$ and $\mathbf{p}$. Accordingly, in the notation of Section 5.1, we take the generalized coordinates for the rigid body to be

$$\mathbf{q} = \begin{bmatrix} \mathbf{r}_{o,I} \\ \mathbf{p} \end{bmatrix} \tag{113}$$

In view of the constraint on $\mathbf{p}$ given by (34), the constraint on $\mathbf{q}$ in (102) becomes

$$\pi(\mathbf{q}) = \Phi(\mathbf{p}) = \mathbf{0}, \tag{114}$$

and equation (104) becomes

$$\Pi(\mathbf{q}) = \begin{bmatrix} \mathbf{0} \ \Xi(\mathbf{p}) \end{bmatrix}, \tag{115}$$

where $\Xi(\mathbf{p})$ is given by (37). By the analysis from section 3.4, this can be equivalently replaced by

$$\Pi(\mathbf{q}) = \begin{bmatrix} \mathbf{0} \ \bar{\Xi}(\mathbf{p}) \end{bmatrix}, \tag{116}$$

where $\bar{\Xi}(\mathbf{p}) \in \mathbb{R}^{(n-3)\times n}$ is any matrix for which (63) is equivalent to (36) and is continuously differentiable on $\mathcal{P}$.

The kinetic energy of the rigid body is readily found to be

$$T = \frac{1}{2}m\mathbf{v}_o^T\mathbf{v}_o + \mathbf{v}_o^T\omega^\times\mathbf{c}_o + \frac{1}{2}\omega^T\mathbf{J}_o\omega, \tag{117}$$

where m is the total mass of the body, $\mathbf{c}_o = \int_B \rho dm$ is the first moment of mass about o and $\mathbf{J}_o = -\int_B \rho^\times \rho^\times dm$ is the moment of inertia about o. There is no potential energy in the problem under consideration, so the Lagrangian is given by

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{v}_o, \omega). \tag{118}$$

It is immediately seen that the Lagrangian in (118) is naturally written as a function of the velocities $\mathbf{v}_o$ and ω, rather than $\dot{\mathbf{r}}_{o,I}$ and $\dot{\mathbf{p}}$. As in (105), we define the generalized velocity

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_o \\ \omega \end{bmatrix}. \tag{119}$$

Making use of (111) and (12), $\alpha(\mathbf{q})$ in (105) becomes

$$\alpha(\mathbf{q}) = \begin{bmatrix} \mathbf{C}(\mathbf{p}) & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{S}}(\mathbf{p}) \end{bmatrix}. \tag{120}$$

Noting that $\mathbf{C}(\mathbf{p})$ is orthonormal for all $\mathbf{p} \in \mathcal{P}$, and making use of (41), $\beta(\mathbf{q})$ in (106) becomes

$$\beta(\mathbf{q}) = \begin{bmatrix} \mathbf{C}^T(\mathbf{p}) & \mathbf{0} \\ \mathbf{0} & \Gamma(\mathbf{p}) \end{bmatrix}. \tag{121}$$

Let us now determine the generalized force $\mathbf{Q}$ corresponding to a virtual displacement $\delta\mathbf{q}$. From (112), we find that the inertial virtual displacement of dm due to (unconstrained) $\delta\mathbf{q}$ is given by

$$\delta\vec{\mathbf{r}}_{dm} = \mathcal{F}_I^T \left[\delta\mathbf{r}_{o,I} + \frac{\partial}{\partial \mathbf{p}^T} \left[\mathbf{C}^T(\mathbf{p})\rho \right] \delta\mathbf{p} \right]. \tag{122}$$

Note, the virtual displacement is executed as if $\mathbf{q}$ were unconstrained [19]; $\mathbf{q}$ remains constrained because of the $\Pi(\mathbf{q})^T\lambda$ term in (108).

Noting that the force acting on dm is $\vec{\mathbf{f}} = \mathcal{F}_b^T \mathbf{f} dV = \mathcal{F}_I^T \mathbf{C}^T(\mathbf{p}) \mathbf{f} dV$, the virtual work performed on dm is

$$\delta W_{dm} = \delta \mathbf{r}_{o,I}^T \mathbf{C}^T \mathbf{f} dV + \delta \mathbf{p}^T \left(\frac{\partial}{\partial \mathbf{p}^T} \left[\mathbf{C}^T(\mathbf{p}) \rho \right] \right)^T \mathbf{C}^T \mathbf{f} dV.$$

Integrating over the body, the total virtual work becomes

$$\delta W = \delta \mathbf{r}_{o,I}^T \mathbf{C}^T \int_B \mathbf{f} dV + \delta \mathbf{p}^T \int_B \left(\frac{\partial}{\partial \mathbf{p}^T} \left[\mathbf{C}^T(\mathbf{p}) \rho \right] \right)^T \mathbf{C}^T \mathbf{f} dV. \tag{123}$$

From this, the generalized force vector is readily identified as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C}^T \mathbf{F} \\ \int_B \left(\frac{\partial}{\partial \mathbf{p}^T} \left[\mathbf{C}^T(\mathbf{p}) \rho \right] \right)^T \mathbf{C}^T \mathbf{f} dV \end{bmatrix}, \tag{124}$$

where $\mathbf{F} = \int_B \mathbf{f} dV$ is the total force acting on the body.

We are now in a position to apply the constrained Boltzmann-Hamel equations given in (109). First, we note that with $\bar{L} = T$, we have $\partial L / \partial \mathbf{q} = \mathbf{0}$, and

$$\frac{\partial \bar{L}}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial T}{\partial \mathbf{v}_o} \\ \frac{\partial T}{\partial \omega} \end{bmatrix}. \tag{125}$$

Next, using (111) and (120), we find that

$$\dot{\alpha}(\mathbf{q}) - \frac{\partial \mathbf{v}}{\partial \mathbf{q}^T} = \begin{bmatrix} \dot{\mathbf{C}}(\mathbf{p}) & -\frac{\partial \mathbf{v}_o}{\partial \mathbf{p}^T} \\ \mathbf{0} & \dot{\bar{\mathbf{S}}}(\mathbf{p}) - \frac{\partial \omega}{\partial \mathbf{p}^T} \end{bmatrix},$$

which upon using (1) and (111) becomes

$$\dot{\alpha}(\mathbf{q}) - \frac{\partial \mathbf{v}}{\partial \mathbf{q}^T} = \begin{bmatrix} -\omega^\times \mathbf{C}(\mathbf{p}) & -\frac{\partial}{\partial \mathbf{p}^T} \left[\mathbf{C}(\mathbf{p}) \dot{\mathbf{r}}_{o,I} \right] \\ \mathbf{0} & \dot{\bar{\mathbf{S}}}(\mathbf{p}) - \frac{\partial \omega}{\partial \mathbf{p}^T} \end{bmatrix}. \tag{126}$$

Using (121) and (126), we obtain

$$\beta(\mathbf{q})^T \left(\dot{\alpha}(\mathbf{q}) - \frac{\partial \mathbf{v}}{\partial \mathbf{q}^T}\right)^T = \begin{bmatrix} \mathbf{C}(\mathbf{p})\mathbf{C}^T(\mathbf{p})\omega^\times & \mathbf{0} \\ -\Gamma^T(\mathbf{p})\left(\frac{\partial}{\partial \mathbf{p}^T}[\mathbf{C}(\mathbf{p})\dot{\mathbf{r}}_{o,I}]\right)^T & \Gamma^T(\mathbf{p})\left(\dot{\bar{\mathbf{S}}}(\mathbf{p}) - \frac{\partial \omega}{\partial \mathbf{p}^T}\right)^T \end{bmatrix}. \tag{127}$$

Using Identity $1'$ in (46), we have

$$\Gamma^T(\mathbf{p})\left(\frac{\partial}{\partial \mathbf{p}^T}[\mathbf{C}(\mathbf{p})\dot{\mathbf{r}}_{o,I}]\right)^T = -(\mathbf{C}(\mathbf{p})\dot{\mathbf{r}}_{o,I})^\times = -\mathbf{v}_o^\times. \tag{128}$$

Therefore, making use of (128) for the bottom left term in (127), the orthonormality of $\mathbf{C}(\mathbf{p})$ for the top left term, and Identity $3'$ in (78) for the bottom right term, we have

$$\beta(\mathbf{q})^T \left(\dot{\alpha}(\mathbf{q}) - \frac{\partial \mathbf{v}}{\partial \mathbf{q}^T}\right)^T = \begin{bmatrix} \omega^\times & \mathbf{0} \\ \mathbf{v}_o^\times & \omega^\times \end{bmatrix}. \tag{129}$$

Let us now evaluate the generalized force term in (109). From (124) and (121), we have

$$\beta(\mathbf{q})^T \mathbf{Q} = \begin{bmatrix} \mathbf{F} \\ \Gamma^T(\mathbf{p}) \int_B \left(\frac{\partial}{\partial \mathbf{p}^T}\left[\mathbf{C}^T(\mathbf{p})\rho\right]\right)^T \mathbf{C}^T \mathbf{f} dV \end{bmatrix}. \tag{130}$$

Noting that $\Gamma(\mathbf{p})$ can be moved inside the integral in (130), Identity $2'$ in (48) can be made use of to reduce the generalized force in (130) to

$$\beta(\mathbf{q})^T \mathbf{Q} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}, \tag{131}$$

where $\mathbf{G} = \int_B \rho^\times \mathbf{f} dV$ is the total external torque acting on the body. Finally, substitution of (125), (129) and (131) into (109) yields the familiar equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \mathbf{v}_o}\right) + \omega^\times \frac{\partial T}{\partial \mathbf{v}_o} = \mathbf{F} \tag{132}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega}\right)+\mathbf{v}_o^\times\frac{\partial T}{\partial \mathbf{v}_o}+\omega^\times\frac{\partial T}{\partial \omega}=\mathbf{G}. \tag{133}$$

If we now take the point o to coincide with the center of mass of the body, then $\mathbf{c}_o=\mathbf{0}$, $\mathbf{J}_o=\mathbf{I}$, and evaluation of the rotational equation (133) becomes the familiar Euler equations

$$\mathbf{I}\dot{\omega}+\omega^\times\mathbf{I}\omega=\mathbf{G}.$$

The derivation presented here remains valid for any minimal (unconstrained) parameter set $\mathbf{p}\in\mathbb{R}^3$, using identities 1 to 3, with $\Gamma(\mathbf{p})$ replaced by $\mathbf{S}^{-1}(\mathbf{p})$. In this case, there is no constraint, and hence no Lagrange multiplier term in the Boltzmann-Hamel equations. The benefit of using a constrained parameter set such as the quaternion or the rotation matrix (as in Section 3.4), is that the derivation holds globally on SO(3), while any minimal parameter set has a singularity which must be avoided for the derivation to be valid.

6 Application: Perturbations in Attitude and the Associated Change in Potential Energy of a Rigid Body

We will now consider how the potential energy of a body changes due to perturbations in attitude. Consider the rigid body in Figure 1. Consider the potential energy associated with the mass element dm:

$$\begin{aligned} dU &= -\vec{\mathbf{g}}\cdot\vec{\mathbf{r}}_{dm}dm \\ &= -g\mathbf{1}_3^T\left(\mathbf{r}_{o,I}+\mathbf{C}^T(\mathbf{p})\rho\right)dm, \end{aligned}$$

where $\mathbf{1}_3=[0\ 0\ 1]^T$, $g=9.81\ m/s^2$, $\vec{\mathbf{r}}_{dm}=\vec{\mathcal{F}}_I^T\left(\mathbf{r}_{o,I}+\mathbf{C}^T(\mathbf{p})\rho\right)$, and $\mathbf{p}$ is associated with any attitude parameterization. The total potential energy is then

$$\begin{aligned} U &= -g\int_B\mathbf{1}_3^T\mathbf{r}_{o,I}dm-g\int_B\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p})\rho dm \\ &= -mg\mathbf{1}_3^T\mathbf{r}_{o,I}-g\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p})\mathbf{c}_o \end{aligned}$$

where $\mathbf{c}_o=\int_B\rho dm$ is the first moment of mass about o. Consider a small perturbation $\delta\mathbf{p}$ resulting in

$$\begin{aligned} U(\mathbf{p}+\delta\mathbf{p}) &= -mg\mathbf{1}_3^T\mathbf{r}_{o,I}-g\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p}+\delta\mathbf{p})\mathbf{c}_o \\ &\approx -mg\mathbf{1}_3^T\mathbf{r}_{o,I}-g\mathbf{1}_3^T\left[\left(\mathbf{1}-(\mathbf{S}(\mathbf{p})\delta\mathbf{p})^\times\right)\mathbf{C}(\mathbf{p})\right]^T\mathbf{c}_o \end{aligned}$$

$$= -mg\mathbf{1}_3^T\mathbf{r}_{o,I} - g\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p})\mathbf{c}_o - g\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p})\left(\mathbf{S}(\mathbf{p})\delta\mathbf{p}\right)^{\times}\mathbf{c}_o$$
$$= -mg\mathbf{1}_3^T\mathbf{r}_{o,I} - g\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p})\mathbf{c}_o + g\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p})\mathbf{c}_o^{\times}\mathbf{S}(\mathbf{p})\delta\mathbf{p}$$

where (90) has been used. Therefore, for small perturbations $\delta\mathbf{p}$ the potential energy changes by $+g\mathbf{1}_3^T\mathbf{C}^T(\mathbf{p})\mathbf{c}_o^{\times}\mathbf{S}(\mathbf{p})\delta\mathbf{p}$.

7 Application: Spacecraft Attitude Determination

We shall now demonstrate the usefulness of the identities developed in this paper for some well-known problems in spacecraft attitude determination. In particular, we examine some aspects of an Extended Kalman Filter (EKF) implementation for the purposes of attitude determination using a minimal (that is, unconstrained) attitude representation. To this end, let $\mathbf{C} \in SO(3)$ be the rotation matrix describing the spacecraft inertial attitude. Specifically, $\mathbf{C}$ transforms vector coordinates from an inertial frame $\mathcal{F}_I$ into a spacecraft body frame $\mathcal{F}_b$. Now, let $\mathbf{p} \in \mathbb{R}^3$ be any minimal (unconstrained) parameterization of $\mathbf{C}$.

Let $\mathbf{x} = \left[\mathbf{p}^T, \mathbf{w}^T\right]^T$ be the state vector to be estimated, with $\mathbf{w} \in \mathbb{R}^n$ containing all other relevant states. For example, $\mathbf{w}$ could contain either or both of angular velocity and gyro bias. As is customary in the development of an EKF, we assume the availability of a deterministic reference state $\bar{\mathbf{x}} = \left[\bar{\mathbf{p}}^T, \bar{\mathbf{w}}^T\right]^T$, such that

$$\mathbf{x} = \bar{\mathbf{x}} + \delta\mathbf{x}, \tag{134}$$

where $\delta\mathbf{x}$ is a small state deviation. The EKF then estimates $\delta\mathbf{x}$ (with estimate denoted by $\delta\hat{\mathbf{x}}$), and the full state estimate is defined as as

$$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \delta\hat{\mathbf{x}}. \tag{135}$$

7.1 Measurement Models

We now consider two types of measurement: vector measurements (such as from magnetometer, digital sun-sensor) and direct attitude measurements (such as from a star-tracker).

Vector Measurements

Vector measurements provide measurements of a reference unit vector (normalized magnetic field vector or sun-vector in the case of magnetometer or digital sun-sensor) in spacecraft body coordinates. These types of measurement take the form:

$$\mathbf{y}_i^v = \mathbf{C}_v(\mathbf{v}_i^v)\mathbf{C}(\mathbf{p})\bar{\mathbf{u}}_{i,I}, \ i = 1, ..., n^v, \tag{136}$$

where $\mathbf{C}_v(\mathbf{v}_i^v)$ is a rotational measurement error (preserving the unit magnitude of the measurement) parameterized by a rotation vector $\mathbf{v}_i^v$, $\mathbf{u}_{i,I}$ is the unit reference vector in inertial coordinates, and n^v is the number of available vector measurements at the sample time under consideration. Note that the form of $\mathbf{C}_v(\mathbf{v}_i^v)$ for a rotation vector is given in equation (95), where ϕ is to be replaced with $\mathbf{v}_i^v$. We use the rotation vector to model the measurement error, since it has units of an angle (in radians), which is a natural choice for representing a rotational error, providing intuitive insight into its meaning.

Corresponding to the measurement in (136), we also define reference vector measurements as

$$\bar{\mathbf{y}}_i^v = \mathbf{C}(\bar{\mathbf{p}})\bar{\mathbf{u}}_{i,I}, \ i = 1, ..., n^v. \tag{137}$$

Assuming small measurement errors $\mathbf{v}_i^v$ (and small deviation $\delta\mathbf{p}$), we expand the measurement in (136) in a first-order Taylor series expansion about $\mathbf{p} = \bar{\mathbf{p}}$, $\mathbf{v}_i^v = \mathbf{0}$. We have

$$\begin{aligned} \mathbf{y}_i^v &= \mathbf{y}_i^v|_{\bar{\mathbf{p}},\mathbf{0}} + \left.\frac{\partial \mathbf{y}_i^v}{\partial \left[\mathbf{p}^T, \mathbf{v}_i^{vT}\right]}\right|_{\bar{\mathbf{p}},\mathbf{0}} \begin{bmatrix} \delta\mathbf{p} \\ \mathbf{v}_i^v \end{bmatrix}, \\ &= \mathbf{C}(\bar{\mathbf{p}})\bar{\mathbf{u}}_{i,I} + \left.\frac{\partial \left(\mathbf{C}_v(\mathbf{v}_i^v)\mathbf{C}(\mathbf{p})\bar{\mathbf{u}}_{i,I}\right)}{\partial \left[\mathbf{p}^T, \mathbf{v}_i^{vT}\right]}\right|_{\bar{\mathbf{p}},\mathbf{0}} \begin{bmatrix} \delta\mathbf{p} \\ \mathbf{v}_i^v \end{bmatrix}. \end{aligned} \tag{138}$$

From Identity 1 in (17), we have

$$\frac{\partial \left(\mathbf{C}_v(\mathbf{v}_i^v)\mathbf{C}(\mathbf{p})\bar{\mathbf{u}}_{i,I}\right)}{\partial \left[\mathbf{p}^T, \mathbf{v}_i^{vT}\right]} = \left(\mathbf{C}_v(\mathbf{v}_i^v)\mathbf{C}(\mathbf{p})\bar{\mathbf{u}}_{i,I}\right)^{\times} \mathbf{S}_{p,v}(\mathbf{p}, \mathbf{v}_i^v), \tag{139}$$

where $\mathbf{S}_{p,v}(\mathbf{p}, \mathbf{v})$ is the kinematic matrix corresponding to the rotation matrix $\mathbf{C}_v(\mathbf{v})\mathbf{C}(\mathbf{p})$, parameterized by $\left[\mathbf{p}^T, \mathbf{v}^T\right]^T$. It is readily shown that this is given by

$$\mathbf{S}_{p,v}(\mathbf{p}, \mathbf{v}) = \begin{bmatrix} \mathbf{C}_v(\mathbf{v})\mathbf{S}(\mathbf{p}) & \mathbf{S}_v(\mathbf{v}) \end{bmatrix}, \tag{140}$$

where $\mathbf{S}_v(\mathbf{v})$ is given by equation (96) with ϕ replaced by $\mathbf{v}$. Consequently, using (137), the vector measurement in (138) becomes

$$\mathbf{y}_i^v = \overline{\mathbf{y}}_i^v + \overline{\mathbf{y}}_i^{v\times}\mathbf{S}(\overline{\mathbf{p}})\delta\mathbf{p} + \overline{\mathbf{y}}_i^{v\times}\mathbf{v}_i^v. \tag{141}$$

Finally, we define the vector measurement deviation as

$$\delta\mathbf{y}_i^v = \mathbf{y}_i^v - \overline{\mathbf{y}}_i^v, \tag{142}$$

such that from (141)

$$\delta\mathbf{y}_i^v = \overline{\mathbf{y}}_i^{v\times}\mathbf{S}(\overline{\mathbf{p}})\delta\mathbf{p} + \overline{\mathbf{v}}_i^v, \tag{143}$$

where the vector measurement noise is given by

$$\overline{\mathbf{v}}_i^v = \overline{\mathbf{y}}_i^{v\times}\mathbf{v}_i^v, \tag{144}$$

which is linear in $\delta\mathbf{p}$ and $\mathbf{v}_i^v$ and can therefore be used as a Kalman Filter input.

Remark: If we had simply taken a first-order Taylor series expansion of (136) about $\mathbf{v}_i^v = \mathbf{0}$, we would obtain

$$\mathbf{y}_i^v = \mathbf{C}(\mathbf{p})\overline{\mathbf{u}}_{i,I} + (\mathbf{C}(\mathbf{p})\overline{\mathbf{u}}_{i,I})^{\times}\,\mathbf{v}_i^v, \tag{145}$$

which is identical in form to the QUEST measurement model presented in [20, 21], but obtained by different means. Note the similarity between the QUEST measurement model (145) and the model presented here (141) (i.e., the noise term has the same form).

Direct Attitude Measurements

These measurements take the form

$$\mathbf{C}(\mathbf{p}^m) = \mathbf{C}_v(\mathbf{v}^a)\mathbf{C}(\mathbf{p}), \tag{146}$$

where $\mathbf{p}^m$ is the measurement of $\mathbf{p}$, and $\mathbf{C}_v(\mathbf{v}^a)$ is a rotational measurement error parameterized by the rotation vector $\mathbf{v}^a$.

Let us define the measured deviation in $\mathbf{p}$ by

$$\delta\mathbf{p}^m = \mathbf{p}^m - \bar{\mathbf{p}} \tag{147}$$

Using the first-order Taylor series expansion in (90), let us now expand both sides of (146) to first order about $\mathbf{p}^m = \mathbf{p} = \bar{\mathbf{p}}$, $\mathbf{v}_i^a = \mathbf{0}$. We have

$$\left(\mathbf{1} - (\mathbf{S}(\bar{\mathbf{p}})\delta\mathbf{p}^m)^\times\right)\mathbf{C}(\bar{\mathbf{p}}) = \left(\mathbf{1} - \left(\mathbf{S}_{p,v}(\bar{\mathbf{p}},\mathbf{0})\begin{bmatrix}\delta\mathbf{p}\\ \mathbf{v}_i^a\end{bmatrix}\right)^\times\right)\mathbf{C}(\bar{\mathbf{p}}).$$

Using (140), we therefore extract

$$\mathbf{S}(\bar{\mathbf{p}})\delta\mathbf{p}^m = \mathbf{S}(\bar{\mathbf{p}})\delta\mathbf{p} + \mathbf{v}_i^a, \tag{148}$$

which can be rearranged to give

$$\delta\mathbf{p}^m = \delta\mathbf{p} + \mathbf{S}^{-1}(\bar{\mathbf{p}})\mathbf{v}_i^a, \tag{149}$$

which is again linear in $\delta\mathbf{p}$ and $\mathbf{v}_i^a$, and can be used as an input for a Kalman Filter.

7.2 Estimation Error

Defining the estimation error by

$$\tilde{\mathbf{x}} := \mathbf{x} - \hat{\mathbf{x}} = \delta\mathbf{x} - \delta\hat{\mathbf{x}}, \tag{150}$$

an EKF implementation returns the covariance matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{pp} & \mathbf{P}_{pw} \\ \mathbf{P}_{pw}^T & \mathbf{P}_{ww} \end{bmatrix}, \tag{151}$$

where $\mathbf{P}_{pp} = E(\tilde{\mathbf{p}}\tilde{\mathbf{p}}^T)$, $\mathbf{P}_{pw} = E(\tilde{\mathbf{p}}\tilde{\mathbf{w}}^T)$ and $\mathbf{P}_{ww} = E(\tilde{\mathbf{w}}\tilde{\mathbf{w}}^T)$, and $E(\cdot)$ denotes the expectation operator.

Let us now define the rotational estimation error Φ, which is a rotation vector, by

$$\mathbf{C}(\mathbf{p}) = \mathbf{C}_v(\Phi)\mathbf{C}(\hat{\mathbf{p}}). \tag{152}$$

This multiplicative definition of estimation error provides more intuition than the additive error $\tilde{\mathbf{p}}$, since it represents a rotation from the estimated to true attitude. In particular, it would be useful to know $\mathbf{P}_{\phi\phi} = E(\Phi\Phi^T)$. We shall now find a first-order relationship between $\mathbf{P}_{pp}$ and $\mathbf{P}_{\phi\phi}$.

To do this, we note from (134) and (135) that (152) can be rewritten as

$$\mathbf{C}(\overline{\mathbf{p}} + \delta\mathbf{p}) = \mathbf{C}_v(\phi)\mathbf{C}(\overline{\mathbf{p}} + \delta\hat{\mathbf{p}}). \tag{153}$$

We now re-use the derivation of (148) to obtain the first-order relationship

$$\mathbf{S}(\overline{\mathbf{p}})\delta\mathbf{p} = \mathbf{S}(\overline{\mathbf{p}})\delta\hat{\mathbf{p}} + \phi, \tag{154}$$

and using (150), this rearranges to give

$$\phi = \mathbf{S}(\overline{\mathbf{p}})\tilde{\mathbf{p}}. \tag{155}$$

Since $\overline{\mathbf{p}}$ is deterministic, we now readily find

$$\mathbf{P}_{\phi\phi} = \mathbf{S}(\overline{\mathbf{p}})\mathbf{P}_{pp}\mathbf{S}^T(\overline{\mathbf{p}}). \tag{156}$$

7.3 Processing of Vector Measurements

Consider the situation where only vector measurements are available. We shall examine the Kalman filter correction step. To this end, let us rewrite (143) as

$$\delta \mathbf{y}_i^v = \mathbf{H}_i^v \delta \mathbf{x} + \overline{\mathbf{v}}_i^v, \tag{157}$$

where

$$\mathbf{H}_i^v = \begin{bmatrix} \overline{\mathbf{y}}_i^{v\times} \mathbf{S}(\overline{\mathbf{p}}) & \mathbf{0} \end{bmatrix}. \tag{158}$$

Suppose now that the vector measurement error $\mathbf{v}_i^v$ is zero-mean (white) noise such that at any sample time, it has covariance

$$E(\mathbf{v}_i^v \mathbf{v}_j^{vT}) = \delta_{ij} r_i^v \mathbf{1}_{3\times 3}, \tag{159}$$

where δ_{ij} is the discrete delta function, and $r_i^v > 0$. Note that this means that the measurement errors are isotropic. Furthermore, this implies that all vector measurements have uncorrelated noises. This means that they may be processed sequentially in the Kalman filter correction step [3]. As such, we consider the correction step for a single vector measurement.

We first note that the additive measurement noise term in (144) is zero-mean, and using (159), its autocovariance is found to be

$$\bar{\mathbf{R}}_i^v = E(\overline{\mathbf{v}}_i^v \overline{\mathbf{v}}_j^{vT}) = -r_i^v \overline{\mathbf{y}}_i^{v\times} \overline{\mathbf{y}}_i^{v\times}. \tag{160}$$

Clearly, $\bar{\mathbf{R}}_i^v$ is only positive semi-definite, with null-space spanned by $\overline{\mathbf{y}}_i^v$. This poses a problem for Kalman filter implementation, since the additive measurement noise term is required to have positive-definite covariance [3]. Returning to (143) and (144), we see that they can be written as

$$\delta \mathbf{y}_i^v = \overline{\mathbf{y}}_i^{v\times} (\mathbf{S}(\overline{\mathbf{p}}) \delta \mathbf{p} + \mathbf{v}_i^v),$$

This shows that the measurement provides no information about $\mathbf{S}(\overline{\mathbf{p}})\delta\mathbf{p} + \mathbf{v}_i^v$ in the direction parallel to $\overline{\mathbf{y}}_i^v$ (since this is in the null-space of $\overline{\mathbf{y}}_i^{v\times}$). Therefore, all useful information contained in $\delta\mathbf{y}_i^v$ may be obtained from its projection onto the plane perpendicular to $\delta\mathbf{y}_i^v$. To this end, let $\mathbf{a}_i^1$ and $\mathbf{a}_i^2$ be unit vectors such that $\mathbf{a}_i^1$, $\mathbf{a}_i^2$ and $\overline{\mathbf{y}}_i^v$ form an orthonormal triad. Then, the appropriate projection of of $\delta\mathbf{y}_i^v$ is given by

$$\delta\mathbf{y}_{i,proj}^v = \begin{bmatrix} \mathbf{a}_i^{1T}\delta\mathbf{y}_i^v \\ \mathbf{a}_i^{2T}\delta\mathbf{y}_i^v \end{bmatrix}, \tag{161}$$

and from (157), we have

$$\delta\mathbf{y}_{i,proj}^v = \mathbf{H}_{i,proj}^v\delta\mathbf{x} + \overline{\mathbf{v}}_{i,proj}^v, \tag{162}$$

where

$$\mathbf{H}_{i,proj}^v = \begin{bmatrix} \mathbf{a}_i^{1T}\overline{\mathbf{y}}_i^{v\times}\mathbf{S}(\overline{\mathbf{p}}) & \mathbf{0} \\ \mathbf{a}_i^{2T}\overline{\mathbf{y}}_i^{v\times}\mathbf{S}(\overline{\mathbf{p}}) & \mathbf{0} \end{bmatrix}, \ \overline{\mathbf{v}}_{i,proj}^v = \begin{bmatrix} \mathbf{a}_i^{1T}\overline{\mathbf{y}}_i^{v\times} \\ \mathbf{a}_i^{2T}\overline{\mathbf{y}}_i^{v\times} \end{bmatrix}\mathbf{v}_i^v. \tag{163}$$

It can now be readily shown that the projected measurement noise term has autocovariance

$$E(\overline{\mathbf{v}}_{i,proj}^v\overline{\mathbf{v}}_{i,proj}^{vT}) = r_i^v\mathbf{1}_{2\times 2}, \tag{164}$$

which is positive-definite. Hence, the projected measurement in (162) can be used in the Kalman filter correction step. Letting $\delta\hat{\mathbf{x}}^-$ and $\mathbf{P}^-$ denote the *a-priori* state estimate and covariance, respectively, the Kalman correction step subsequently takes the form

$$\Delta\mathbf{x} = \mathbf{K}_{proj}\left(\delta\mathbf{y}_{i,proj}^v - \mathbf{H}_{i,proj}^v\delta\hat{\mathbf{x}}^-\right), \tag{165}$$

for the state and

$$\mathbf{P}^+ = \left(\mathbf{1} - \mathbf{K}_{proj}\mathbf{H}_{i,proj}^v\right)\mathbf{P}^-\left(\mathbf{1} - \mathbf{K}_{proj}\mathbf{H}_{i,proj}^v\right)^T + r_i^v\mathbf{K}_{proj}\mathbf{K}_{proj}^T, \tag{166}$$

for the covariance, where $\mathbf{P}^+$ is the *a-posteriori* covariance, and the gain is given by

$$\mathbf{K}_{proj} = \mathbf{P}^- \mathbf{H}_{i,proj}^{vT} \left(\mathbf{H}_{i,proj}^{v} \mathbf{P}^- \mathbf{H}_{i,proj}^{vT} + r_i^v \mathbf{1}_{2\times 2} \right)^{-1}. \tag{167}$$

Note that the a-posteriori state estimate is given by $\delta\hat{\mathbf{x}}^+ = \delta\hat{\mathbf{x}}^- + \Delta\mathbf{x}$.

Using the Sherman-Morrison-Woodbury matrix inversion lemma [3], the correction step defined by (165) to (167) is equivalently given by

$$\Delta\mathbf{x} = \frac{1}{r_i^v} \mathbf{P}^+ \mathbf{H}_{i,proj}^{vT} \left(\delta\mathbf{y}_{i,proj}^{v} - \mathbf{H}_{i,proj}^{v} \delta\hat{\mathbf{x}}^- \right), \tag{168}$$

and

$$\mathbf{P}^+ = \left((\mathbf{P}^-)^{-1} + \frac{1}{r_i^v} \mathbf{H}_{i,proj}^{vT} \mathbf{H}_{i,proj}^{v} \right)^{-1}. \tag{169}$$

Now, by (162), the state correction in (168) can be written as

$$\Delta\mathbf{x} = \frac{1}{r_i^v} \mathbf{P}^+ \mathbf{H}_{i,proj}^{vT} \mathbf{H}_{i,proj}^{v} \left(\delta\mathbf{x} - \delta\hat{\mathbf{x}}^- \right) + \frac{1}{r_i^v} \mathbf{P}^+ \mathbf{H}_{i,proj}^{vT} \overline{\mathbf{v}}_{i,proj}^{v}. \tag{170}$$

Now, using (163), we evaluate

$$\mathbf{H}_{i,proj}^{vT} \mathbf{H}_{i,proj}^{v} = \begin{bmatrix} -\mathbf{S}(\overline{\mathbf{p}})^T \overline{\mathbf{y}}_i^{v\times} \left(\mathbf{a}_i^1 \mathbf{a}_i^{1T} + \mathbf{a}_i^2 \mathbf{a}_i^{2T} \right) \overline{\mathbf{y}}_i^{v\times} \mathbf{S}(\overline{\mathbf{p}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Since $\overline{\mathbf{y}}_i^{v\times} \overline{\mathbf{y}}_i^{v} = \mathbf{0}$, we may write

$$\overline{\mathbf{y}}_i^{v\times} \left(\mathbf{a}_i^1 \mathbf{a}_i^{1T} + \mathbf{a}_i^2 \mathbf{a}_i^{2T} \right) \overline{\mathbf{y}}_i^{v\times} = \overline{\mathbf{y}}_i^{v\times} \left(\mathbf{a}_i^1 \mathbf{a}_i^{1T} + \mathbf{a}_i^2 \mathbf{a}_i^{2T} + \overline{\mathbf{y}}_i^{v} \overline{\mathbf{y}}_i^{vT} \right) \overline{\mathbf{y}}_i^{v\times},$$

and since $\mathbf{a}_i^1$ and $\mathbf{a}_i^2$ and $\overline{\mathbf{y}}_i^v$ form an orthornormal triad, it follows that

$$\mathbf{H}_{i,proj}^{vT}\mathbf{H}_{i,proj}^{v} = \begin{bmatrix} -\mathbf{S}(\overline{\mathbf{p}})^T\overline{\mathbf{y}}_i^{v\times}\overline{\mathbf{y}}_i^{v\times}\mathbf{S}(\overline{\mathbf{p}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Using (158), we easily find that

$$\mathbf{H}_{i,proj}^{vT}\mathbf{H}_{i,proj}^{v} = \mathbf{H}_i^{vT}\mathbf{H}_i^{v}. \tag{171}$$

In a similar manner, we readily find that

$$\mathbf{H}_{i,proj}^{vT}\begin{bmatrix} \mathbf{a}_i^{1T}\overline{\mathbf{y}}_i^{v\times} \\ \mathbf{a}_i^{2T}\overline{\mathbf{y}}_i^{v\times} \end{bmatrix} = \mathbf{H}_i^{vT}\overline{\mathbf{y}}_i^{v\times}, \tag{172}$$

which means from (144) and (163) that

$$\mathbf{H}_{i,proj}^{vT}\overline{\mathbf{v}}_{i,proj}^{v} = \mathbf{H}_i^{vT}\overline{\mathbf{v}}_i^{v}. \tag{173}$$

Therefore, the state and autocovariance corrections in (170) and (169) may be written equivalently as

$$\Delta\mathbf{x} = \frac{1}{r_i^v}\mathbf{P}^+\mathbf{H}_i^{vT}\mathbf{H}_i^{v}\left(\delta\mathbf{x} - \delta\hat{\mathbf{x}}^-\right) + \frac{1}{r_i^v}\mathbf{P}^+\mathbf{H}_i^{vT}\overline{\mathbf{v}}_i^{v}. \tag{174}$$

and

$$\mathbf{P}^+ = \left((\mathbf{P}^-)^{-1} + \frac{1}{r_i^v}\mathbf{H}_i^{vT}\mathbf{H}_i^{v}\right)^{-1}. \tag{175}$$

Using (157), we see that the state correction in (174) is equivalently given by

$$\Delta\mathbf{x} = \frac{1}{r_i^v}\mathbf{P}^+\mathbf{H}_i^{vT}\left(\delta\mathbf{y}_i^v - \mathbf{H}_i^v\delta\hat{\mathbf{x}}^-\right), \tag{176}$$

Finally, using the Sherman-Morrison-Woodbury matrix inversion lemma [3] again, we can rewrite the correction step in (175) and (176) back in the form given in equations (165) to (167) as

$$\Delta\mathbf{x} = \mathbf{K}\left(\delta\mathbf{y}_i^v - \mathbf{H}_i^v\delta\hat{\mathbf{x}}^-\right), \tag{177}$$

$$\mathbf{P}^+ = \left(\mathbf{1} - \mathbf{K}\mathbf{H}_i^v\right)\mathbf{P}^-\left(\mathbf{1} - \mathbf{K}\mathbf{H}_i^v\right)^T + r_i^v\mathbf{K}\mathbf{K}^T, \tag{178}$$

and

$$\mathbf{K} = \mathbf{P}^-\mathbf{H}_i^{vT}\left(\mathbf{H}_i^v\mathbf{P}^-\mathbf{H}_i^{vT} + r_i^v\mathbf{1}_{3\times3}\right)^{-1}. \tag{179}$$

Therefore, in the special case that all vector measurements are isotropic and uncorrelated with each other, as well as being uncorrelated with the cosine measurement noises, then we can use the full vector measurements in the Kalman filter by setting $E(\bar{\mathbf{v}}_i^v\bar{\mathbf{v}}_i^{vT}) = r_i^v\mathbf{1}_{3\times3}$ where $r_i^v\mathbf{1}_{3\times3} = E(\mathbf{v}_i^v\mathbf{v}_i^{vT})$ for each vector measurement. Although not statistically correct, it yields statistically equivalent state and covariance updates. This result was first presented in [21] for the case of a Multiplicative EKF. Using the identities derived in this paper, we have shown that this remains true for any additive EKF for attitude determination when a minimal attitude parameterization is used.

8 Concluding Remarks

Although rotation matrices represent attitude both globally and uniquely, parameterizations are often favoured and used for a variety of reasons. This paper considers various identities that are very useful when, for example, deriving the motion equations of a rigid-body using Lagrange's equation, or formulating attitude determination problems. In particular, we derive six identities, the first three relevant to unconstrained parameterizations, and the last three relevant to constrained parameterizations. We also consider rotation matrix perturbations. We present various examples, highlighting the utility of our results.

The results presented herein are applicable to many problems found in engineering, making the results not only novel, but also practically relevant. Although the proof of the six identities and the development of the perturbation results are mathematically elegant, the value and significance of our work lies in their simple application to practical, common, real-world engineering problems found in mechanical engineering, aerospace engineering, robotics, control systems design, and estimation.

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List of table captions

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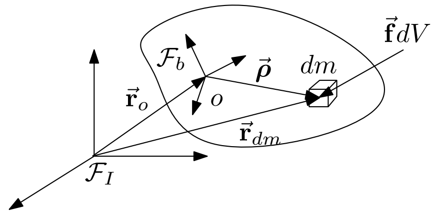
List of figure captions

Figure 1: Free-body diagram of a rigid body



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