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Generalized Euler Sequences Revisited

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Abstract It has previously been shown in the literature that the generalized Euler sequences (also known as Davenport sequences) provide a universal set of attitude representations. However, while these works assert the existence of generalized Euler angles for a given attitude, they do not provide explicit ranges that those angles must lie within. In addition, they do not generally provide physical insight into what a generalized Euler sequence is. This paper addresses these two points. As such, this paper contains a comprehensive self-contained treatment of generalized Euler sequences. In particular, a constructive approach is taken to proving that the generalized Euler sequences provide a universal set of attitude representations. In doing so, specific ranges that contain the generalized Euler angles are obtained, and physical insight is provided into the generalized Euler sequences. The singularity of the generalized Euler sequences is characterized, and it is shown that the generalized Euler angles are uniquely specified within their restricted ranges when away from the singularity condition.

Keywords Generalized Euler Angles · Davenport Angles · Euler Sequences

Introduction

The set of all rotation matrices describing the rotational transformation between two reference frames is given by the special orthogonal group, $SO(3)$, which consists of all real 3×3 orthonormal matrices with determinant $+1$. Leonard Euler stated and proved that any element of $SO(3)$ (any rotation

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matrix) could be decomposed into three successive rotations about principal axes, provided no successive rotations are about the same axis [3]. That is, these sequences provide universal representations of $SO(3)$. These sequences are called Euler rotation sequences, and the corresponding angles are called Euler angles. There are twelve possible distinct Euler sequences [6].

Davenport sought to generalize the Euler sequence to three successive rotations about arbitrary axes with no successive axes parallel [2]. Davenport [2] and later Shuster and Markley [8] and Wittenburg and Lilov [9] proved that such a rotation sequence is only able to represent any element of $SO(3)$ if the first and second axes of rotation are perpendicular, and the second and third axes of rotation are perpendicular. These types of rotation sequences are called generalized Euler sequences or Davenport sequences, and corresponding angles are called generalized Euler angles or Davenport angles. The book [5] also provides a treatment of generalized Euler sequences.

The references [2,8,9,5] provide separate and different proofs that the generalized Euler sequences provide universal representations of $SO(3)$. In particular, in all of the aforementioned articles, for a given element of $SO(3)$, the existence of corresponding generalized Euler angles is proven. However, prior literature do not obtain explicit ranges containing such angles. Furthermore, the proofs presented in the aforementioned papers do not provide a physical insight of what a generalized Euler sequence is, and why it provides a universal representation of $SO(3)$.

In this paper generalized Euler sequences are revisited. Like the existing literature we prove that the generalized Euler sequences provide universal representations of $SO(3)$. However, unlike the previous literature, we explicitly construct a generalized Euler sequence for a given element of $SO(3)$ with the corresponding angles restricted to appropriately defined ranges. In doing so we obtain a stronger result than what exists in the literature in that the generalized Euler sequences, with angles restricted to appropriately defined ranges, provide a universal representation of $SO(3)$. As an additional benefit, the constructive approach taken in this paper is then used to provide physical insight into what a generalized Euler sequence is. Having established the existence of generalized Euler angles within appropriately defined ranges, we then proceed to obtain uniqueness conditions for the angles when restricted to those ranges, as well as characterizing the singularity condition associated with the generalized Euler sequence. The singularity condition has been previously presented in [8], however, we include it for completeness as well as the fact that it follows naturally from the uniqueness result.

For clarity and pedagogy we have attempted to make this paper as self contained and complete as possible. To this end, we first present some well-known preliminary results. Following this, we present the existence and uniqueness results for the generalized Euler angles, as well as a characterization of the associated singularity. The method of proof is completely algebraic, without the need for any physical arguments. Next, we provide a physical interpretation of the generalized Euler sequence, which parallels the algebraic construction in the proof of existence. We also provide a physical interpretation of the singularity condition. Finally, for completeness we include a short treatment of the generalized Euler angle kinematics.

Notation and Preliminary Results

We first describe the notation used in this paper. The Euclidean 2-norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted by $\|\mathbf{x}\|$. The identity matrix is denoted by $\mathbf{1}$, where the dimension will be implied by context. The unit sphere embedded in $\mathbb{R}^3$ will be denoted by $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$. For physical interpretations of rotations, physical vectors will be used. In particular, physical vectors will be denoted by an overarrow , such as $\vec{\mathbf{v}}$. The relationship between a physical vector, the physical basis vectors composing a reference frame, and the components of a physical vector resolved in a reference frame is [4]

$$\begin{aligned}\vec{\mathbf{v}} &= \vec{\mathbf{i}}_a(v_{a1}) + \vec{\mathbf{j}}_a(v_{a2}) + \vec{\mathbf{k}}_a(v_{a3}) \\ &= \begin{bmatrix} \vec{\mathbf{i}}_a & \vec{\mathbf{j}}_a & \vec{\mathbf{k}}_a \end{bmatrix} \begin{bmatrix} v_{a1} \\ v_{a2} \\ v_{a3} \end{bmatrix} \\ &= \vec{\mathcal{F}}_a^T \mathbf{v}_a\end{aligned}$$

where $\vec{\mathcal{F}}_a^T = [\vec{\mathbf{i}}_a \quad \vec{\mathbf{j}}_a \quad \vec{\mathbf{k}}_a]$ is a vectrix [4] containing the unit physical basis vectors defining frame $\mathcal{F}_a$ and $\mathbf{v}_a = [v_{a1} \quad v_{a2} \quad v_{a3}]^T$ is a vector composed of the components of $\vec{\mathbf{v}}$ resolved in the reference frame $\mathcal{F}_a$. We denote the standard orthonormal basis for $\mathbb{R}^3$ by $\mathbf{e}_1 = [1\ \ 0\ \ 0]^T$, $\mathbf{e}_2 = [0\ \ 1\ \ 0]^T$ and $\mathbf{e}_3 = [0\ \ 0\ \ 1]^T$.

The set describing all rotations in three-dimensions is given by

$$SO(3) = \{\mathbf{C} \in \mathbb{R}^{3\times3} : \mathbf{C}^T\mathbf{C} = \mathbf{1},\ \det \mathbf{C} = +1\}.$$

It is straightforward to verify that if $\mathbf{C}_1, \mathbf{C}_2 \in SO(3)$, then so is $\mathbf{C}_3 = \mathbf{C}_2\mathbf{C}_1$. In addition, if $\mathbf{C} \in SO(3)$, then $\mathbf{C}^{-1} = \mathbf{C}^T \in SO(3)$. Finally, $\mathbf{1} \in SO(3)$.

Consider the unit vector $\mathbf{a} \in S^2$, and angle $\phi \in \mathbb{R}$, with corresponding matrix

$$\mathbf{C}_a(\phi) = \cos\phi\mathbf{1} + (1 - \cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi\mathbf{a}^\times, \tag{1}$$

for any $\mathbf{a} = [a_1\ \ a_2\ \ a_3]^T \in \mathbb{R}^3$, where we define

$$\mathbf{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Notice that $\mathbf{C}_a^T(\phi)\mathbf{C}_a(\phi) = \mathbf{1}$ follows by direct multiplication and application of Lemma 2 below, which implies that $\det \mathbf{C}_a(\phi) = \pm 1$. Additionally, $\det \mathbf{C}_a(\phi) = 1$ follows by the fact that $\det \mathbf{C}_a(0) = 1$ and continuity of $\det \mathbf{C}_a(\phi)$. Therefore, $\mathbf{C}_a(\phi) \in SO(3)$.

We shall make frequent use of a number of facts. First, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\mathbf{a}^\times\mathbf{b} = -\mathbf{b}^\times\mathbf{a}$.

Lemma 1 ([1],[4]) *For any* $\mathbf{C} \in SO(3)$ *and any* $\mathbf{a} \in \mathbb{R}^3$, $(\mathbf{Ca})^\times = \mathbf{Ca}^\times\mathbf{C}^T$.

Lemma 2 ([1],[4]) *For any* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$\mathbf{a}^\times\mathbf{b}^\times = \mathbf{ba}^T - (\mathbf{a}^T\mathbf{b})\mathbf{1}.$$

Lemma 3 ([1]) *For any* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$,

$$\mathbf{a}^T\mathbf{b}^\times\mathbf{c} = \mathbf{c}^T\mathbf{a}^\times\mathbf{b}.$$

The following properties of $\mathbf{C}_a(\phi)$ in (1) are readily verified directly from its definition, utilizing the above Lemmas, and some common trigonometric identities.

Lemma 4 *Consider* $\mathbf{C}_a(\phi)$, *as defined in (1). The following facts hold:*

1. $$\mathbf{C}_a(\phi)\mathbf{a} = \mathbf{C}_a(\phi)^T\mathbf{a} = \mathbf{a},$$
   for all $\phi \in \mathbb{R}$.
2. $$\mathbf{C}_a(-\phi) = \mathbf{C}_{-a}(\phi) = \mathbf{C}_a(\phi)^T,$$
   for any $\phi \in \mathbb{R}$.
3. $$\mathbf{C}_a(\pi) = \mathbf{C}_a(\pi)^T = \mathbf{C}_a(-\pi) = \mathbf{C}_{-a}(\pi) = -\mathbf{1} + 2\mathbf{aa}^T.$$
4. *Let* $\mathbf{b} = \mathbf{Ma}$, *where* $\mathbf{M} \in SO(3)$, *and* $\mathbf{a} \in S^2$. *Then,*
   $$\mathbf{MC}_a(\phi)\mathbf{M}^T = \mathbf{C}_b(\phi),$$
   for any $\phi \in \mathbb{R}$.
5. $$\mathbf{C}_a(\phi_1)\mathbf{C}_a(\phi_2) = \mathbf{C}_a(\phi_2)\mathbf{C}_a(\phi_1) = \mathbf{C}_a(\phi_1 + \phi_2),$$
   for any $\phi_1, \phi_2 \in \mathbb{R}$.
6. *Suppose that* $\mathbf{b} \in S^2$ *and* $\mathbf{b}^T\mathbf{a} = 0$. *Then,*
   $$\mathbf{C}_a(\pi)\mathbf{b} = -\mathbf{b}.$$

Proposition 1 *Suppose that* $\mathbf{a}, \mathbf{b} \in S^2$. *Then,* $\mathbf{a} = \mathbf{b}$ *if and only if* $\mathbf{a}^T\mathbf{b} = 1$.

Proof By the Cauchy-Schwarz inequality [1], $|\mathbf{a}^T\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$, with equality if and only if $\mathbf{a} = \lambda\mathbf{b}$, for some $\lambda \in \mathbb{R}$. Therefore, $\mathbf{a}^T\mathbf{b} = 1$ implies that $\mathbf{a} = \pm\mathbf{b}$. However, only $\mathbf{a} = \mathbf{b}$ yields $\mathbf{a}^T\mathbf{b} = 1$.

We shall now present a purely algebraic proof of Euler's rotational theorem [4], which will be needed in the subsequent analysis. While Euler's Theorem is well known, the algebraic proof is interesting in and of itself, since it does not rely on any physical or geometric interpretations of $\mathbf{C} \in SO(3)$. In addition, giving the proof allows us to state a corollary which will be needed later in the paper.

Theorem 1 (Euler's Rotational Theorem) *Given any* $\mathbf{C} \in SO(3)$*, it is possible to find* $\mathbf{a} \in S^2$ *and* $\phi \in (-\pi, \pi]$ *such that* $\mathbf{C} = \mathbf{C}_a(\phi)$*.*

Proof The proof is constructive. Consider any $\mathbf{C} \in SO(3)$. Then,

$$\begin{aligned}\det[\mathbf{C} - \mathbf{1}] &= \det \mathbf{C}^T \det[\mathbf{C} - \mathbf{1}],\\ &= \det[\mathbf{C}^T(\mathbf{C} - \mathbf{1})]\\ &= \det[\mathbf{1} - \mathbf{C}^T]\\ &= \det[\mathbf{1} - \mathbf{C}]\\ &= -\det[\mathbf{C} - \mathbf{1}],\end{aligned}$$

which implies that $\det[\mathbf{C} - \mathbf{1}] = 0$, and consequently $\mathbf{C}$ has eigenvalue $+1$. Let $\mathbf{a} \in S^2$ be a corresponding unit eigenvector, such that

$$\mathbf{C}\mathbf{a} = \mathbf{a}.$$

Now, consider any $\mathbf{b} \in S^2$, such that $\mathbf{a}^T\mathbf{b} = 0$, and define $\mathbf{c} = \mathbf{a}^\times\mathbf{b}$. Then, the matrix

$$\mathbf{M} = \begin{bmatrix}\mathbf{a} \; \mathbf{b} \; \mathbf{c}\end{bmatrix} \in SO(3).$$

In particular, we note that

$$\mathbf{a} = \mathbf{M}\mathbf{e}_1. \tag{2}$$

Furthermore,

$$\mathbf{C}\mathbf{M} = \begin{bmatrix}\mathbf{a} \; \mathbf{d} \; \mathbf{e}\end{bmatrix},$$

where $\mathbf{d} = \mathbf{C}\mathbf{b} \in S^2$ and $\mathbf{e} = \mathbf{C}\mathbf{c} \in S^2$. Next,

$$\mathbf{M}^T\mathbf{C}\mathbf{M} = \begin{bmatrix}\mathbf{e}_1 \; \mathbf{f} \; \mathbf{g}\end{bmatrix}, \tag{3}$$

for some $\mathbf{f}, \mathbf{g} \in S^2$. Noting that $SO(3)$ is closed under matrix multiplication, it follows that $\mathbf{f}^T\mathbf{e}_1 = 0$. Consequently, $\mathbf{f} \in \operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3\}$. Noting that $\mathbf{e}_3 = \mathbf{e}_1^\times\mathbf{e}_2$, it follows that $\mathbf{f}$ can be written as

$$\mathbf{f} = \cos\phi\mathbf{e}_2 - \sin\phi\mathbf{e}_1^\times\mathbf{e}_2 = \mathbf{C}_{e_1}(\phi)\mathbf{e}_2, \tag{4}$$

for some $\phi \in (-\pi, \pi]$. Note that the fact that $\mathbf{e}_1^T\mathbf{e}_2 = 0$ has been used to obtain the second equality in (4). Consequently, from (4) we have

$$\mathbf{C}_{e_1}^T(\phi)\mathbf{f} = \mathbf{C}_{e_1}^T(\phi)\mathbf{C}_{e_1}(\phi)\mathbf{e}_2 = \mathbf{e}_2. \tag{5}$$

Therefore, utilizing equations (3), (5) and Lemma 4 part 1, we obtain

$$\mathbf{C}_{e_1}^T(\phi)\mathbf{M}^T\mathbf{C}\mathbf{M} = \begin{bmatrix}\mathbf{e}_1 \; \mathbf{e}_2 \; \mathbf{C}_{e_1}^T(\phi)\mathbf{g}\end{bmatrix}.$$

Again, since $SO(3)$ is closed under matrix multiplication, it must be that $\mathbf{C}_{e_1}^T(\phi)\mathbf{g} = \mathbf{e}_3$. Therefore,

$$\mathbf{C}_{e_1}^T(\phi)\mathbf{M}^T\mathbf{C}\mathbf{M} = \mathbf{1}.$$

Rearranging gives

$$\mathbf{C} = \mathbf{M}\mathbf{C}_{e_1}(\phi)\mathbf{M}^T = \mathbf{C}_a(\phi),$$

where equation (2) and Lemma 4 part 4 has been used. This completes the proof.

Corollary 1 *Consider any* $\mathbf{C} \in SO(3)$*. Suppose that* $\mathbf{a} \in S^2$ *satisfies* $\mathbf{Ca} = \mathbf{a}$*. Then, there exists* $\phi \in (-\pi, \pi]$ *such that* $\mathbf{C} = \mathbf{C}_a(\phi)$*.*

Proof This follows directly from the proof of Theorem 1.

Generalized Euler Sequences

Consider three axes $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in S^2$, such that $\mathbf{a}_1^T \mathbf{a}_2 = \mathbf{a}_2^T \mathbf{a}_3 = 0$. Then, a generalized Euler sequence is given by the matrix [2,8,5]

$$\mathbf{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \phi_1, \phi_2, \phi_3) = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1), \tag{6}$$

where $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$. Since $SO(3)$ is closed under matrix multiplication as noted above, it follows that $\mathbf{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \phi_1, \phi_2, \phi_3) \in SO(3)$. Note that when $\mathbf{a}_i \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then the generalized Euler sequence reduces to a standard Euler sequence [6]. As noted in the introduction, Davenport [2] and later Shuster and Markley [8] and Wittenburg and Lilov [9] proved that the orthogonality condition $\mathbf{a}_1^T \mathbf{a}_2 = \mathbf{a}_2^T \mathbf{a}_3 = 0$ is necessary for the rotation sequence in (6) to be able to represent any element of $SO(3)$. For completeness, we include a proof of this necessity, following the approach given in [5].

Theorem 2 *A necessary condition for the generalized Euler sequence in (6) to be able to represent any element in* $SO(3)$ *is that* $\mathbf{a}_1^T \mathbf{a}_2 = \mathbf{a}_2^T \mathbf{a}_3 = 0$*.*

Proof To prove this, we first note that given any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ with $\|\mathbf{x}\| = \|\mathbf{y}\| \neq 0$, there exists $\mathbf{C} \in SO(3)$ such that $\mathbf{y} = \mathbf{Cx}$. To see why, as in the proof of Theorem 1, select $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in S^2$ such that $\mathbf{A} = \begin{bmatrix} \mathbf{x}/\|\mathbf{x}\| & \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 \end{bmatrix} \in SO(3)$ and $\mathbf{B} = \begin{bmatrix} \mathbf{y}/\|\mathbf{y}\| & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 \end{bmatrix} \in SO(3)$. Then, $\mathbf{x} = \|\mathbf{x}\|\mathbf{A}\mathbf{e}_1$ and $\mathbf{y} = \|\mathbf{y}\|\mathbf{B}\mathbf{e}_1$. Setting $\mathbf{C} = \mathbf{B}\mathbf{A}^T \in SO(3)$, one has $\mathbf{y} = \mathbf{Cx}$. Now, set $\mathbf{x} = \mathbf{a}_1$ and $\mathbf{y} = \mathbf{a}_3$, such that $\mathbf{a}_3 = \mathbf{C}\mathbf{a}_1$ for some $\mathbf{C} \in SO(3)$. If the generalized Euler sequence is able to represent any element in $SO(3)$, then it must be possible to find angles $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$ such that $\mathbf{a}_3 = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{a}_1$. Pre-multiplying this by $\mathbf{C}_{a_3}(\phi_3)^T$ and using Lemma 4 part 1, gives $\mathbf{a}_3 = \mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1$. Pre-multiplying again by $\mathbf{a}_2^T$ and expanding using (1) leads to

$$\mathbf{a}_2^T \mathbf{a}_3 = \mathbf{a}_2^T \mathbf{a}_1. \tag{7}$$

Next, set $\mathbf{x} = \mathbf{a}_1$ and $\mathbf{y} = -\mathbf{a}_3$, such that $-\mathbf{a}_3 = \bar{\mathbf{C}}\mathbf{a}_1$ for some $\bar{\mathbf{C}} \in SO(3)$. In a similar manner as before, this leads upon simplification to

$$-\mathbf{a}_2^T \mathbf{a}_3 = \mathbf{a}_2^T \mathbf{a}_1. \tag{8}$$

Combining (7) and (8), we obtain $\mathbf{a}_2^T \mathbf{a}_3 = \mathbf{a}_2^T \mathbf{a}_1 = 0$. This completes the proof.

Existence of the Generalized Euler Angles

We now state and prove an existence result for the generalized Euler angles.

Theorem 3 *Consider any generalized Euler sequence, and let* $\alpha \in (-\pi, \pi]$ *be the unique angle in* $(-\pi, \pi]$ *satisfying*

$$\sin\alpha = \mathbf{a}_3^T \mathbf{a}_1^\times \mathbf{a}_2, \;\; \cos\alpha = \mathbf{a}_3^T \mathbf{a}_1. \tag{9}$$

Then:

1. *The mapping* $\mathbf{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \phi_1, \phi_2, \phi_3) : (-\pi, \pi] \times [\alpha, \pi + \alpha] \times (-\pi, \pi] \to SO(3)$ *is surjective.*
2. *The mapping* $\mathbf{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \phi_1, \phi_2, \phi_3) : (-\pi, \pi] \times [\alpha - \pi, \alpha] \times (-\pi, \pi] \to SO(3)$ *is surjective.*

The implication of this statement is that, given any $\mathbf{C} \in SO(3)$, for generalized Euler sequences it is possible to find angles $\phi_1, \phi_3 \in (-\pi, \pi]$ and $\phi_2 \in [\alpha, \pi+\alpha]$ such that $\mathbf{C} = \mathbf{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \phi_1, \phi_2, \phi_3)$. Likewise, it is also possible to find angles $\phi_1, \phi_3 \in (-\pi, \pi]$ and $\phi_2 \in [\alpha-\pi, \alpha]$ such that $\mathbf{C} = \mathbf{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \phi_1, \phi_2, \phi_3)$

Proof The proof will be constructive. We will first show where the expressions given in equation (9) come from. Recall that for generalized Euler sequences, $\mathbf{a}_1^T \mathbf{a}_2 = 0$. Hence, $\mathbf{a}_1, \mathbf{a}_2^\times \mathbf{a}_1, \mathbf{a}_2$ forms an orthonormal basis for $\mathbb{R}^3$. Next, since for generalized Euler sequences $\mathbf{a}_2^T \mathbf{a}_3 = 0$, it follows that $\mathbf{a}_3 \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2^\times \mathbf{a}_1\}$. Furthermore, since $\mathbf{a}_3 \in S^2$, it therefore follows that $\mathbf{a}_3$ can be written as

$$\mathbf{a}_3 = \cos\alpha \mathbf{a}_1 - \sin\alpha \mathbf{a}_2^\times \mathbf{a}_1 = \mathbf{C}_{a_2}(\alpha)\mathbf{a}_1, \tag{10}$$

for some $\alpha \in \mathbb{R}$. Note that the fact that $\mathbf{a}_1^T \mathbf{a}_2 = 0$ has been used to obtain the second equality in (4). From (10), it follows directly that

$$\sin\alpha = \mathbf{a}_3^T \mathbf{a}_1^\times \mathbf{a}_2, \;\; \cos\alpha = \mathbf{a}_3^T \mathbf{a}_1, \tag{11}$$

which uniquely specifies α in $(-\pi, \pi]$.

Now consider any $\mathbf{C} \in SO(3)$. We shall now proceed to construct a generalized Euler sequence such that $\mathbf{C} = \mathbf{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \phi_1, \phi_2, \phi_3)$, where $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times [\alpha, \pi + \alpha] \times (-\pi, \pi]$ for Case 1 of the Theorem, and $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times [\alpha - \pi, \alpha] \times (-\pi, \pi]$ for Case 2 of the Theorem.

Step 1
Suppose for the moment that the given $\mathbf{C} \in SO(3)$ is representable by a generalized Euler sequence as in equation (6). Then,

$$\mathbf{C}^T = \mathbf{C}_{a_1}^T(\phi_1)\mathbf{C}_{a_2}^T(\phi_2)\mathbf{C}_{a_3}^T(\phi_3). \tag{12}$$

Pre- and post-multiplying both sides of equation (12) by $\mathbf{a}_2^T \mathbf{C}_{a_1}(\phi_1)$ and $\mathbf{a}_3$, respectively, and using Lemma 4 part 1 and the fact that $\mathbf{a}_2^T \mathbf{a}_3 = 0$ gives

$$\mathbf{a}_2^T \mathbf{C}_{a_1}(\phi_1)\mathbf{C}^T \mathbf{a}_3 = 0. \tag{13}$$

We therefore seek the first rotation angle $\phi_1 \in (-\pi, \pi]$ such that equation (13) holds. Expanding equation (13) leads to

$$\begin{aligned} 0 &= \mathbf{a}_2^T \left(\cos\phi_1 \mathbf{1} + (1 - \cos\phi_1)\mathbf{a}_1\mathbf{a}_1^T - \sin\phi_1 \mathbf{a}_1^\times \right) \mathbf{C}^T\mathbf{a}_3, \\ &= \cos\phi_1 \mathbf{a}_2^T\mathbf{C}^T\mathbf{a}_1 - \sin\phi_1 \mathbf{a}_2^T\mathbf{a}_1^\times\mathbf{C}^T\mathbf{a}_3, \end{aligned}$$

where the fact that $\mathbf{a}_2^T\mathbf{a}_1 = 0$ has been used. This now leads to

$$\cos\phi_1 \mathbf{a}_2^T\mathbf{C}^T\mathbf{a}_3 = \sin\phi_1 \mathbf{a}_2^T\mathbf{a}_1^\times\mathbf{C}^T\mathbf{a}_3. \tag{14}$$

There are now four cases to consider for (14).

1. $\mathbf{a}_2^T\mathbf{C}^T\mathbf{a}_3 \neq 0$ and $\mathbf{a}_2^T\mathbf{a}_1^\times\mathbf{C}^T\mathbf{a}_3 \neq 0$. In this case, it is not possible to have $\cos\phi_1 = 0$ (since this would simultaneously require that $\sin\phi_1 = 0$, which is impossible). Hence, this leads to
$$\tan\phi_1 = \frac{\mathbf{a}_2^T\mathbf{C}^T\mathbf{a}_3}{\mathbf{a}_2^T\mathbf{a}_1^\times\mathbf{C}^T\mathbf{a}_3},$$
which has a unique solution on $(-\pi/2, \pi/2)$. Call this solution $\bar{\phi}_1 \in (-\pi/2, \pi/2)$. Furthermore, $\tan\phi = \tan(\phi \pm \pi)$, so there is an additional solution in $(-\pi, -\pi/2) \cup (\pi/2, \pi)$, given by
$$\phi_1' = \begin{cases} \bar{\phi}_1 + \pi, \ \bar{\phi}_1 < 0, \\ \bar{\phi}_1 - \pi, \ \bar{\phi}_1 > 0. \end{cases}$$
Hence, in this case, there is a pair of solutions for ϕ_1 inside $(-\pi, \pi) \subset (-\pi, \pi]$, separated by π.
2. $\mathbf{a}_2^T\mathbf{C}^T\mathbf{a}_3 = 0$ and $\mathbf{a}_2^T\mathbf{a}_1^\times\mathbf{C}^T\mathbf{a}_3 \neq 0$. In this case, $\sin\phi_1 = 0$, for which there are two solutions inside $(-\pi, \pi]$, which are given by $\bar{\phi}_1 = 0$ and $\phi_1' = \pi$. Hence, in this case, there is a pair of solutions for ϕ_1 inside $(-\pi, \pi]$, separated by π.
3. $\mathbf{a}_2^T\mathbf{C}^T\mathbf{a}_3 \neq 0$ and $\mathbf{a}_2^T\mathbf{a}_1^\times\mathbf{C}^T\mathbf{a}_3 = 0$. In this case, $\cos\phi_1 = 0$, for which there are two solutions inside $(-\pi, \pi]$, which are given by $\bar{\phi}_1 = -\pi/2$ and $\phi_1' = \pi/2$. Hence, in this case, there is a pair of solutions for ϕ_1 inside $(-\pi, \pi]$, separated by π.
4. $\mathbf{a}_2^T\mathbf{C}^T\mathbf{a}_3 = \mathbf{a}_2^T\mathbf{a}_1^\times\mathbf{C}^T\mathbf{a}_3 = 0$. In this case, any $\phi_1 \in (-\pi, \pi]$ is a solution. In particular, it is possible to find a pair of solutions for ϕ_1 inside $(-\pi, \pi]$, separated by π.

Hence, in all cases, there are a pair of solutions to (13) for ϕ_1 inside $(-\pi, \pi]$, separated by π, such that they can be written as $\bar{\phi}_1$ and $\phi_1' = \bar{\phi}_1 + a\pi$, where $a \in \{-1, 1\}$. Let us now examine the two solutions further. Making use of the trigonometric identities $\cos(x \pm \pi) = -\cos x$ and $\sin(x \pm \pi) = -\sin x$, we have

$$\begin{aligned} \mathbf{C}_{a_1}(\phi_1')\mathbf{C}^T\mathbf{a}_3 &= \mathbf{C}_{a_1}(\bar{\phi}_1 \pm \pi)\mathbf{C}^T\mathbf{a}_3, \\ &= \left(-\cos\bar{\phi}_1 \mathbf{1} + (1 + \cos\bar{\phi}_1)\mathbf{a}_1\mathbf{a}_1^T + \sin\bar{\phi}_1 \mathbf{a}_1^\times \right) \mathbf{C}^T\mathbf{a}_3, \\ &= -\left(\cos\bar{\phi}_1 \mathbf{1} + (1 - \cos\bar{\phi}_1)\mathbf{a}_1\mathbf{a}_1^T - \sin\bar{\phi}_1 \mathbf{a}_1^\times \right) \mathbf{C}^T\mathbf{a}_3 + 2\mathbf{a}_1\mathbf{a}_1^T\mathbf{C}^T\mathbf{a}_3, \\ &= -\mathbf{C}_{a_1}(\bar{\phi}_1)\mathbf{C}^T\mathbf{a}_3 + 2\mathbf{a}_1\mathbf{a}_1^T\mathbf{C}_{a_1}(\bar{\phi}_1)\mathbf{C}^T\mathbf{a}_3, \end{aligned} \tag{15}$$

where Lemma 4 part 1 has been used to obtain the second term on right hand side of equation (15). We therefore obtain

$$\mathbf{C}_{a_1}(\phi_1')\mathbf{C}^T\mathbf{a}_3 = \left(-\mathbf{1} + 2\mathbf{a}_1\mathbf{a}_1^T\right)\mathbf{C}_{a_1}(\bar{\phi}_1)\mathbf{C}^T\mathbf{a}_3. \tag{16}$$

Hence, the two resulting vectors $\mathbf{C}_{a_1}(\bar{\phi}_1)\mathbf{C}^T\mathbf{a}_3$ and $\mathbf{C}_{a_1}(\phi_1')\mathbf{C}^T\mathbf{a}_3$ are reflections of each other about $\mathbf{a}_1$. Equivalently, from Lemma 4 part 3, they are related to each other by a rotation of $\pm\pi$ around $\mathbf{a}_1$.

Step 2
Let $\phi_1 \in (-\pi, \pi]$ denote either angle from Step 1 (either $\bar{\phi}$ or ϕ_1'). Now, we wish to choose $\phi_2 \in [\alpha, \pi + \alpha]$ in Case 1, and $\phi_2 \in [\alpha - \pi, \alpha]$ in Case 2, such that

$$\mathbf{a}_3 = \mathbf{C}_{a_2}(\phi_2)\mathbf{w}, \tag{17}$$

where $\mathbf{w} = \mathbf{C}_{a_1}(\phi_1)\mathbf{C}^T\mathbf{a}_3$ from Step 1. Note that there are two possibilities for $\mathbf{w}$ from Step 1, namely $\bar{\mathbf{w}} = \mathbf{C}_{a_1}(\bar{\phi}_1)\mathbf{C}^T\mathbf{a}_3$ and $\mathbf{w}' = \mathbf{C}_{a_1}(\phi_1')\mathbf{C}^T\mathbf{a}_3$.

Since $\mathbf{w}$ has unit length, and is perpendicular to $\mathbf{a}_2$ (from equation (13) in Step 1), similarly to $\mathbf{a}_3$ in (10), $\mathbf{w}$ can be written

$$\mathbf{w} = \cos(\phi - \alpha)\mathbf{a}_1 + \sin(\phi - \alpha)\mathbf{a}_2^{\times}\mathbf{a}_1 = \mathbf{C}_{a_2}^T(\phi - \alpha)\mathbf{a}_1, \tag{18}$$

for some $\phi \in \mathbb{R}$ with $(\phi - \alpha) \in [-\pi, \pi]$. From (18), we have

$$\cos(\phi - \alpha) = \mathbf{a}_1^T\mathbf{w}, \;\; \sin(\phi - \alpha) = \mathbf{w}^T\mathbf{a}_2^{\times}\mathbf{a}_1. \tag{19}$$

Setting $\phi_2 = \phi$ where ϕ satisfies (19), we obtain from (18), (10) and Lemma 4 parts 2 and 5, that

$$\begin{aligned}\mathbf{C}_{a_2}(\phi_2)\mathbf{w} &= \mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_2}^T(\phi_2 - \alpha)\mathbf{a}_1,\\ &= \mathbf{C}_{a_2}(\alpha)\mathbf{a}_1,\\ &= \mathbf{a}_3,\end{aligned}$$

as required.

Next we will show that it is possible to choose $\phi_2 \in [\alpha, \pi + \alpha]$ for Case 1 of the Theorem and $\phi_2 \in [\alpha - \pi, \alpha]$ for Case 2 of the Theorem. For Case 1, we must have $(\phi_2 - \alpha) \in [0, \pi]$, which implies that $\sin(\phi - \alpha) \geq 0$. From (19), this leads to the requirement that $\mathbf{w}^T\mathbf{a}_2^{\times}\mathbf{a}_1 \geq 0$. For Case 2, we must have $(\phi_2 - \alpha) \in [-\pi, 0]$, which implies that $\sin(\phi - \alpha) \leq 0$. From (19), this leads to the requirement that $\mathbf{w}^T\mathbf{a}_2^{\times}\mathbf{a}_1 \leq 0$. Recall that from Step 1, there were at least two solutions for ϕ_1, separated by π. We will now show that one of these solutions always results in the desired sign of $\mathbf{w}^T\mathbf{a}_2^{\times}\mathbf{a}_1$. From equation (16) we define $\mathbf{w}' = (-\mathbf{1} + 2\mathbf{e}_i\mathbf{e}_i^T)\bar{\mathbf{w}}$. It follows that

$$\begin{aligned}(\mathbf{a}_2^{\times}\mathbf{a}_1)^T\mathbf{w}' &= (\mathbf{a}_2^{\times}\mathbf{a}_1)^T\left(-\mathbf{1} + 2\mathbf{a}_1\mathbf{a}_1^T\right)\bar{\mathbf{w}},\\ &= -(\mathbf{a}_2^{\times}\mathbf{a}_1)^T\bar{\mathbf{w}} + 2(\mathbf{a}_2^{\times}\mathbf{a}_1)^T\mathbf{a}_1\mathbf{a}_1^T\bar{\mathbf{w}}.\end{aligned}$$

Noting that $(\mathbf{a}_2^{\times}\mathbf{a}_1)^T\mathbf{a}_1 = 0$, this becomes

$$(\mathbf{a}_2^{\times}\mathbf{a}_1)^T\mathbf{w}' = -(\mathbf{a}_2^{\times}\mathbf{a}_1)^T\bar{\mathbf{w}}. \tag{20}$$

Therefore, from (20), it is always possible to choose ϕ_1 from Step 1 such that $\mathbf{w}^T\mathbf{a}_2^\times\mathbf{a}_1 \geq 0$ for Case 1, which therefore results in $\phi_2 \in [\alpha, \pi+\alpha]$. Likewise, it is always possible to choose ϕ_1 from Step 1 such that $\mathbf{w}^T\mathbf{a}_2^\times\mathbf{a}_1 \leq 0$, which therefore results in $\phi_2 \in [\alpha-\pi, \alpha]$.

To summarize the results so far, we have shown that it is always possible to find $\phi_1 \in (-\pi, \pi]$ and $\phi_2 \in [\alpha, \pi+\alpha]$, such that

$$\mathbf{a}_3 = \mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{C}^T\mathbf{a}_3. \tag{21}$$

Likewise, it is always possible to find $\phi_1 \in (-\pi, \pi]$ and $\phi_2 \in [\alpha-\pi, \alpha]$, such that again (21) holds.

Step 3
Since $\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{C}^T \in SO(3)$, it follows from (21) and Corollary 1 that

$$\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{C}^T = \mathbf{C}_{a_3}(-\bar{\phi}),$$

for some $\bar{\phi} \in (-\pi, \pi]$. Noting that $\mathbf{C}_{a_3}(\pi) = \mathbf{C}_{a_3}(-\pi)$, we now take

$$\phi_3 = \begin{cases} \bar{\phi}, & \bar{\phi} \neq \pi, \\ -\pi, & \bar{\phi} = \pi. \end{cases}$$

Consequently, we have

$$\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{C}^T = \mathbf{C}_{a_3}(-\phi_3),$$

which leads to

$$\mathbf{C} = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1),$$

with either $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times [\alpha, \pi+\alpha] \times (-\pi, \pi]$, as required for Case 1, or $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times [\alpha-\pi, \alpha] \times (-\pi, \pi]$, as required for Case 2. This completes the proof.

Let us briefly consider some special cases of generalized Euler sequences.

1. Consider the case when the first and third axes are equal ($\mathbf{a}_1 = \mathbf{a}_3$), as occurs with standard symmetric Euler sequences [6]. In this case, equation (11) becomes $\sin\alpha = 0$, $\cos\alpha = 1$, which leads to $\alpha = 0$. Consequently, for these types of generalized Euler sequences, given any $\mathbf{C} \in SO(3)$, it is possible to find $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times [0, \pi] \times (-\pi, \pi]$ such that $\mathbf{C} = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)$. Likewise, given any $\mathbf{C} \in SO(3)$, it is possible to find $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times [-\pi, 0] \times (-\pi, \pi]$ such that $\mathbf{C} = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)$.
2. Consider when the first and third axes are orthogonal ($\mathbf{a}_1 \perp \mathbf{a}_3$), as occurs with standard anti-symmetric Euler sequences [6]. In this case, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form an orthonormal basis for $\mathbb{R}^3$, and in particular, $\mathbf{a}_3 = a\mathbf{a}_1^\times\mathbf{a}_2$, where $a \in \{-1, 1\}$. In this case, equation (11) becomes $\sin\alpha = a$, $\cos\alpha = 0$, which leads to either $\alpha = -\pi/2$, or $\alpha = \pi/2$. Consequently, in Case 1, the range for ϕ_2 either becomes $[-\pi/2, \pi/2]$ or $[\pi/2, 3\pi/2]$. Likewise, in Case 2, the range for ϕ_2 becomes either $[-3\pi/2, -\pi/2]$ or $[-\pi/2, \pi/2]$.

Now, both the ranges $[\pi/2, 3\pi/2]$ and $[-3\pi/2, -\pi/2]$ are equivalent to the range $[-\pi, -\pi/2] \cup [\pi/2, \pi]$. As such, for these types of generalized Euler sequences, given any $\mathbf{C} \in SO(3)$, it is possible to find $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times [-\pi/2, \pi/2] \times (-\pi, \pi]$ such that $\mathbf{C} = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)$. Likewise, given any $\mathbf{C} \in SO(3)$, it is possible to find $(\phi_1, \phi_2, \phi_3) \in (-\pi, \pi] \times ([-\pi, -\pi/2] \cup [\pi/2, \pi]) \times (-\pi, \pi]$ such that $\mathbf{C} = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)$.

Uniqueness of the Generalized Euler Angles

Given the existence result in Theorem 3, we shall determine how to obtain the generalized Euler angles ϕ_1, ϕ_2, ϕ_3 within the given ranges from $\mathbf{C} \in SO(3)$.

Step 1: Determining ϕ_2
Utilizing Lemma 4 parts 1, 2 and 5, together with equation (10), we obtain

$$\begin{aligned}
\mathbf{a}_3^T\mathbf{C}\mathbf{a}_1 &= \mathbf{a}_3^T\mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{a}_1, \\
&= \mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1, \\
&= \mathbf{a}_1^T\mathbf{C}_{a_2}^T(\alpha)\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1, \\
&= \mathbf{a}_1^T\mathbf{C}_{a_2}(\phi_2 - \alpha)\mathbf{a}_1.
\end{aligned} \tag{22}$$

Upon expansion of the right-hand side of (22) we have

$$\mathbf{a}_3^T\mathbf{C}\mathbf{a}_1 = \cos(\phi_2 - \alpha), \tag{23}$$

which may be uniquely solved for ϕ_2 on the interval $\phi_2 \in [\alpha, \pi + \alpha]$ owing to the fact that, from Theorem 3, such a ϕ_2 exists. Likewise, equation (23) may be uniquely solved for ϕ_2 on the interval $\phi_2 \in [\alpha - \pi, \alpha]$ where, again, such a ϕ_2 exists from Theorem 3. Furthermore, let $\bar{\phi}_2$ and ϕ_2' be the two solutions of (23) in the intervals $[\alpha, \pi + \alpha]$ and $[\alpha - \pi, \alpha]$, respectively. Then, by (23), we have

$$\phi_2' - \alpha = -(\bar{\phi}_2 - \alpha), \tag{24}$$

which shows that the two solutions are related by

$$\phi_2' = 2\alpha - \bar{\phi}_2. \tag{25}$$

Step 2: Determining ϕ_1
Having determined ϕ_2 in Step 1 (either of the two solutions), $\mathbf{C}_{a_2}(\phi_2)$ may readily be computed. We shall now compute a corresponding solution for ϕ_1. First, utilizing Lemma 4 part 1, compute

$$\begin{aligned}
\mathbf{a}_3^T\mathbf{C}\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3 &= \mathbf{a}_3^T\mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3, \\
&= \mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3, \\
&= \mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\left(\cos\phi_1\mathbf{1} + (1-\cos\phi_1)\mathbf{a}_1\mathbf{a}_1^T - \sin\phi_1\mathbf{a}_1^\times\right)\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3, \\
&= \cos\phi_1 + (1-\cos\phi_1)(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2, \\
&= (\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 + \cos\phi_1(1 - (\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2).
\end{aligned}$$

Provided that $(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 \neq 1$, this can be solved for $\cos\phi_1$ as

$$\cos\phi_1 = \frac{\mathbf{a}_3^T\mathbf{C}\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3 - (\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2}{1-(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2}. \tag{26}$$

Likewise, utilizing Lemma 4 part 1 and Lemma 2, compute

$$\begin{aligned}\mathbf{a}_3^T\mathbf{C}\mathbf{a}_1^\times\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3 &= \mathbf{a}_3^T\mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{a}_1^\times\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3,\\ &= \mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)\mathbf{a}_1^\times\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3,\\ &= \mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\left(\cos\phi_1\mathbf{1}+(1-\cos\phi_1)\mathbf{a}_1\mathbf{a}_1^T-\sin\phi_1\mathbf{a}_1^\times\right)\mathbf{a}_1^\times\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3,\\ &= -\sin\phi_1\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1^\times\mathbf{a}_1^\times\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3,\\ &= \sin\phi_1(1-(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2).\end{aligned}$$

Again, provided $(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 \neq 1$, this can be solved for $\sin\phi_1$ as

$$\sin\phi_1 = \frac{\mathbf{a}_3^T\mathbf{C}\mathbf{a}_1^\times\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3}{1-(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2}. \tag{27}$$

Consequently, if $(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 \neq 1$, equations (26) and (27) may be uniquely solved for ϕ_1 on the interval $\phi_1 \in (-\pi,\pi]$. The condition $(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 = 1$ corresponds to a singularity of the generalized Euler sequence, and will be dealt with separately in a later section.

Step 3: Determining ϕ_3
We shall now likewise compute a solution for ϕ_3, corresponding to ϕ_2 from Step 1, in a similar manner to ϕ_1 from step 1. First, compute

$$\mathbf{a}_1^T\mathbf{C}_{a_2}^T(\phi_2)\mathbf{C}\mathbf{a}_1 = (\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 + \cos\phi_3(1-(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2),$$

and if $(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 \neq 1$, this can be solved for $\cos\phi_3$ as

$$\cos\phi_3 = \frac{\mathbf{a}_1^T\mathbf{C}_{a_2}^T(\phi_2)\mathbf{C}\mathbf{a}_1 - (\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2}{1-(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2}. \tag{28}$$

Likewise,

$$\mathbf{a}_1^T\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3^\times\mathbf{C}\mathbf{a}_1 = \sin\phi_3(1-(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2),$$

and if $(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 \neq 1$, this can be solved for $\sin\phi_3$ as

$$\sin\phi_3 = \frac{\mathbf{a}_1^T\mathbf{C}_{a_2}^T(\phi_2)\mathbf{a}_3^\times\mathbf{C}\mathbf{a}_1}{1-(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2}. \tag{29}$$

Consequently, if $(\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2 \neq 1$, equations (28) and (29) may be uniquely solved for ϕ_3 on the interval $\phi_3 \in (-\pi,\pi]$.

Note from (22) that $\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1 = \mathbf{a}_3^T\mathbf{C}\mathbf{a}_1$, and therefore, ϕ_1 and ϕ_3 can be uniquely solved for from Steps 2 and 3 if $\|\mathbf{a}_3^T\mathbf{C}\mathbf{a}_1\| \neq 1$.

We now summarize our findings thus far in a theorem.

Theorem 4 *Consider any generalized Euler sequence, with* $\alpha \in (-\pi, \pi]$ *as specified in Theorem 3. Consider any* $\mathbf{C} \in SO(3)$*, and let* $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \in (-\pi, \pi] \times [\alpha, \pi + \alpha] \times (-\pi, \pi]$*, and* $(\phi_1', \phi_2', \phi_3') \in (-\pi, \pi] \times [\alpha - \pi, \alpha] \times (-\pi, \pi]$ *satisfy*

$$\mathbf{C} = \mathbf{C}_{a_3}(\bar{\phi}_3)\mathbf{C}_{a_2}(\bar{\phi}_2)\mathbf{C}_{a_1}(\bar{\phi}_1) = \mathbf{C}_{a_3}(\phi_3')\mathbf{C}_{a_2}(\phi_2')\mathbf{C}_{a_1}(\phi_1'). \tag{30}$$

Then, $\bar{\phi}_2$ *is uniquely specified in* $[\alpha, \pi + \alpha]$*, and* ϕ_2' *is uniquely specified in* $[\alpha - \pi, \alpha]$*. In addition, if* $\|\mathbf{a}_3^T \mathbf{C} \mathbf{a}_1\| \neq 1$*, then* $\bar{\phi}_1$*,* $\bar{\phi}_3$ *and* ϕ_1'*,* ϕ_3' *are uniquely specified in* $(-\pi, \pi]$ *also.*

Together, Theorems 3 and 4 provide an existence and uniqueness result for the generalized Euler angles, when they are restricted to appropriate ranges.

As a by-product of obtaining the uniqueness result in Theorem 4, we have obtained explicit expressions for the determination of the angles ϕ_1, ϕ_2 and ϕ_3 within their specified ranges. Both [8] and [7] provide alternative expressions for determining the angles, which may be more efficient in practical applications. Theorems 3 and 4 in this paper provide rigorous justification that the expressions, either in this paper, or those in [8] or [7], may be applied to find the generalized Euler angles within the ranges specified in Theorems 3 and 4 (that is, such angles exist).

Relationship Between the Two Representations

We now examine the relationship between the two generalized Euler sequence representations for a given $\mathbf{C} \in SO(3)$, as given in (30), under the condition $\|\mathbf{a}_3^T \mathbf{C} \mathbf{a}_1\| \neq 1$, such that the generalized Euler angles are unique within their specified ranges. This examination is similar to the one given in [5], except that we explicitly enforce the requirement that the two representations lie in their given ranges.

We already have the relationship between $\bar{\phi}_2$ and ϕ_2', which is given in (25) (or equivalently in (24)). Next, utilizing equation (10) and Lemma 4 part 4, we have

$$\mathbf{C}_{a_3}(\bar{\phi}_3) = \mathbf{C}_{a_2}(\alpha)\mathbf{C}_{a_1}(\bar{\phi}_3)\mathbf{C}_{a_2}^T(\alpha). \tag{31}$$

Consequently, utilizing Lemma 4 parts 2 and 5,

$$\begin{aligned} \mathbf{C} &= \mathbf{C}_{a_3}(\bar{\phi}_3)\mathbf{C}_{a_2}(\bar{\phi}_2)\mathbf{C}_{a_1}(\bar{\phi}_1), \\ &= \mathbf{C}_{a_2}(\alpha)\mathbf{C}_{a_1}(\bar{\phi}_3)\mathbf{C}_{a_2}^T(\alpha)\mathbf{C}_{a_2}(\bar{\phi}_2)\mathbf{C}_{a_1}(\bar{\phi}_1), \\ &= \mathbf{C}_{a_2}(\alpha)\mathbf{C}_{a_1}(\bar{\phi}_3)\mathbf{C}_{a_2}(\bar{\phi}_2 - \alpha)\mathbf{C}_{a_1}(\bar{\phi}_1). \end{aligned} \tag{32}$$

Now, let $b, c \in \{-1, 1\}$. Then, using Lemma 4 parts 3 and 5, (32) becomes

$$\begin{aligned} \mathbf{C} &= \mathbf{C}_{a_2}(\alpha)\mathbf{C}_{a_1}(\bar{\phi}_3)\mathbf{C}_{a_1}(b\pi)\mathbf{C}_{a_1}(\pi)\mathbf{C}_{a_2}(\bar{\phi}_2 - \alpha)\mathbf{C}_{a_1}(\pi)\mathbf{C}_{a_1}(c\pi)\mathbf{C}_{a_1}(\bar{\phi}_1), \\ &= \mathbf{C}_{a_2}(\alpha)\mathbf{C}_{a_1}(\bar{\phi}_3 + b\pi)\mathbf{C}_{a_1}(\pi)\mathbf{C}_{a_2}(\bar{\phi}_2 - \alpha)\mathbf{C}_{a_1}(\pi)\mathbf{C}_{a_1}(\bar{\phi}_1 + c\pi). \end{aligned} \tag{33}$$

Next, since $\mathbf{a}_1^T\mathbf{a}_2 = 0$, Lemma 4 part 6 gives $\mathbf{C}_{a_1}(\pi)\mathbf{a}_2 = -\mathbf{a}_2$. Consequently, utilizing Lemma 4 parts 2, 3, and 4, and equation (24)

$$\begin{aligned}\mathbf{C}_{a_1}(\pi)\mathbf{C}_{a_2}(\bar{\phi}_2 - \alpha)\mathbf{C}_{a_1}(\pi) &= \mathbf{C}_{-a_2}(\bar{\phi}_2 - \alpha), \\ &= \mathbf{C}_{a_2}(\alpha - \bar{\phi}_2), \\ &= \mathbf{C}_{a_2}(\phi_2' - \alpha). \end{aligned} \tag{34}$$

Substituting (34) into (33) and utilizing Lemma 4 parts 2, 4 and 5, we obtain

$$\begin{aligned}\mathbf{C} &= \mathbf{C}_{a_2}(\alpha)\mathbf{C}_{a_1}(\bar{\phi}_3 + b\pi)\mathbf{C}_{a_2}(\phi_2' - \alpha)\mathbf{C}_{a_1}(\bar{\phi}_1 + c\pi), \\ &= \mathbf{C}_{a_2}(\alpha)\mathbf{C}_{a_1}(\bar{\phi}_3 + b\pi)\mathbf{C}_{a_2}^T(\alpha)\mathbf{C}_{a_2}(\phi_2')\mathbf{C}_{a_1}(\bar{\phi}_1 + c\pi), \\ &= \mathbf{C}_{a_3}(\bar{\phi}_3 + b\pi)\mathbf{C}_{a_2}(\phi_2')\mathbf{C}_{a_1}(\bar{\phi}_1 + c\pi). \end{aligned} \tag{35}$$

Finally, since $\bar{\phi}_1, \bar{\phi}_3 \in (-\pi, \pi]$, it must be that one of $\bar{\phi}_1 \pm \pi$ is in the range $(-\pi, \pi]$, and one of $\bar{\phi}_3 \pm \pi$ is in the range $(-\pi, \pi]$. Therefore, since ϕ_1' and ϕ_3' are uniquely specified in the range $(-\pi, \pi]$, comparing (35) with (30) gives

$$\phi_1' = \begin{cases} \bar{\phi}_1 - \pi, \ \bar{\phi}_1 > 0, \\ \bar{\phi}_1 + \pi, \ \bar{\phi}_1 \le 0, \end{cases}, \quad \phi_3' = \begin{cases} \bar{\phi}_3 - \pi, \ \bar{\phi}_3 > 0, \\ \bar{\phi}_3 + \pi, \ \bar{\phi}_3 \le 0, \end{cases} \tag{36}$$

Singularity of the Generalized Euler Sequence

Let us now further examine the condition when the angles ϕ_1 and ϕ_3 are not uniquely solvable for within their specified ranges. From the previous subsection, this occurs when

$$\mathbf{a}_3^T\mathbf{C}\mathbf{a}_1 = \mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1 = \pm 1. \tag{37}$$

We shall call this condition a singularity of $\mathbf{C} = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)$. From (23), we see that the singularity condition is equivalently given by

$$\cos(\phi_2 - \alpha) = \pm 1, \tag{38}$$

which implies that $\phi_2 - \alpha = 0, \pm\pi$. Consequently, the singularity condition becomes

$$\phi_2 = \alpha, \alpha \pm \pi. \tag{39}$$

Note is that the singularity condition depends on the angle ϕ_2 only. Let us briefly consider some special cases of generalized Euler sequences.

1. Consider when the first and third axes are equal ($\mathbf{a}_1 = \mathbf{a}_3$), as occurs with symmetric standard Euler sequences [6]. In this case, $\alpha = 0$, and the singularity condition reduces to

$$\phi_2 = 0, \pm\pi.$$

2. Consider the case were the first and third axes are orthogonal ($\mathbf{a}_1 \perp \mathbf{a}_3$), as occurs with anti-symmetric standard Euler sequences [6]. In this case, $\alpha = \pm\pi/2$. Consequently, the singularity condition becomes

$$\phi_2 = \pm\pi/2.$$

Next, utilizing Proposition 1, the singularity condition (37) implies that

$$\mathbf{a}_3 = \pm\mathbf{C}\mathbf{a}_1 = \pm\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1. \tag{40}$$

Consequently, utilizing Lemma 4 parts 2 and 4 and (40), $\mathbf{C}_{a_3}(\phi_3)$ becomes

$$\begin{aligned}\mathbf{C}_{a_3}(\phi_3) &= \mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{\pm a_1}(\phi_3)\mathbf{C}_{a_2}^T(\phi_2),\\ &= \mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\pm\phi_3)\mathbf{C}_{a_2}^T(\phi_2).\end{aligned}$$

Using this result we have that

$$\begin{aligned}\mathbf{C} &= \mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\pm\phi_3)\mathbf{C}_{a_2}^T(\phi_2)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1),\\ &= \mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\pm\phi_3)\mathbf{C}_{a_1}(\phi_1),\end{aligned}$$

which by Lemma 4 part 5 gives

$$\mathbf{C} = \mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1 \pm \phi_3). \tag{41}$$

This shows that at the singularity, the rotation sequence has effectively been reduced to two rotations, one with angle ϕ_2, and the other with angle $\phi_1 \pm \phi_3$. In this case, we have

$$\begin{aligned}\mathbf{a}_2^T\mathbf{C}\mathbf{a}_2 &= \mathbf{a}_2^T\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1 \pm \phi_3)\mathbf{a}_2,\\ &= \mathbf{a}_2^T\mathbf{C}_{a_1}(\phi_1 \pm \phi_3)\mathbf{a}_2,\\ &= \cos(\phi_1 \pm \phi_3).\end{aligned} \tag{42}$$

Likewise,

$$\begin{aligned}\mathbf{a}_2^T\mathbf{C}\mathbf{a}_1^\times\mathbf{a}_2 &= \mathbf{a}_2^T\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1 \pm \phi_3)\mathbf{a}_1^\times\mathbf{a}_2,\\ &= \mathbf{a}_2^T\mathbf{C}_{a_1}(\phi_1 \pm \phi_3)\mathbf{a}_1^\times\mathbf{a}_2,\\ &= \mathbf{a}_2^T\left(\cos(\phi_1 \pm \phi_3)\mathbf{1} + (1-\cos(\phi_1 \pm \phi_3)\mathbf{a}_1\mathbf{a}_1^T - \sin(\phi_1 \pm \phi_3)\mathbf{a}_1^\times\right)\mathbf{a}_1^\times\mathbf{a}_2,\\ &= \sin(\phi_1 \pm \phi_3),\end{aligned}$$

where Lemma 2 has been used. Together, equations (42) and (43) may be solved uniquely for $\phi_1 \pm \phi_3$, restricted to an appropriate interval of length 2π. Correspondingly, there are infinite solutions for ϕ_1 and ϕ_3, on the interval $(-\pi, \pi]$.

Physical Interpretation of the Generalized Euler Sequence

The developments up to this point have been purely algebraic, without any physical or geometric interpretation. We shall now provide a physical interpretation of the generalized Euler sequence, paralleling the construction in the proof of Theorem 3.

Associated with a generalized Euler sequence, consider four frames $\mathcal{F}_a$, $\mathcal{F}_q$, $\mathcal{F}_r$ and $\mathcal{F}_b$ and three axes $\vec{\mathbf{a}}_1 = \vec{\mathcal{F}}_a^T \mathbf{a}_1$, $\vec{\mathbf{a}}_2 = \vec{\mathcal{F}}_q^T \mathbf{a}_2$ and $\vec{\mathbf{a}}_3 = \vec{\mathcal{F}}_r^T \mathbf{a}_3$, such that the four frames are related by the successive rotations

$$\mathcal{F}_a \overset{\mathbf{C}_{a_1}(\phi_1)}{\rightarrow} \mathcal{F}_q \overset{\mathbf{C}_{a_2}(\phi_2)}{\rightarrow} \mathcal{F}_r \overset{\mathbf{C}_{a_3}(\phi_3)}{\rightarrow} \mathcal{F}_b.$$

Now, associated with each of the frames $\mathcal{F}_a$, $\mathcal{F}_q$ and $\mathcal{F}_r$, define the following triads

$$\vec{\mathbf{a}}_1^x = \vec{\mathcal{F}}_x^T \mathbf{a}_1, \;\; \vec{\mathbf{a}}_2^x = \vec{\mathcal{F}}_x^T \mathbf{a}_2, \;\; \vec{\mathbf{a}}_3^x = \vec{\mathcal{F}}_x^T \mathbf{a}_3, \text{ for }, x = a, q, r.$$

Note that $\vec{\mathbf{a}}_1^x, \vec{\mathbf{a}}_3^x \perp \vec{\mathbf{a}}_2^x$, for $x = a, q, r$. We embed these triads in each of the respective frames, such that under the successive generalized Euler rotations we have

$$(\vec{\mathbf{a}}_1^a, \vec{\mathbf{a}}_2^a, \vec{\mathbf{a}}_3^a) \overset{\mathbf{C}_{a_1}(\phi_1)}{\rightarrow} (\vec{\mathbf{a}}_1^q, \vec{\mathbf{a}}_2^q, \vec{\mathbf{a}}_3^q) \overset{\mathbf{C}_{a_2}(\phi_2)}{\rightarrow} (\vec{\mathbf{a}}_1^r, \vec{\mathbf{a}}_2^r, \vec{\mathbf{a}}_3^r).$$

In particular, note that $\vec{\mathbf{a}}_1 = \vec{\mathbf{a}}_1^a = \vec{\mathbf{a}}_1^q$, and $\vec{\mathbf{a}}_2^q = \vec{\mathbf{a}}_2^r$. Now, if the first two rotations can be performed, such that $\vec{\mathbf{a}}_3^r = \vec{\mathbf{a}}_3$, then $\vec{\mathbf{a}}_3$ would be the principal axis of rotation between frames $\mathcal{F}_r$ and $\mathcal{F}_b$ (since it has the same coordinate representation in both frames). In this case, the third angle $\phi_3 \in (-\pi, \pi]$ can be found, such that rotating $\mathcal{F}_r$ about $\vec{\mathbf{a}}_3$ through angle ϕ_3 yields $\mathcal{F}_b$. Consequently, the purpose of the first two rotations in the generalized Euler sequence is simply to rotate the triad $(\vec{\mathbf{a}}_1^a, \vec{\mathbf{a}}_2^a, \vec{\mathbf{a}}_3^a)$ to $(\vec{\mathbf{a}}_1^q, \vec{\mathbf{a}}_2^q, \vec{\mathbf{a}}_3^q)$ and then to $(\vec{\mathbf{a}}_1^r, \vec{\mathbf{a}}_2^r, \vec{\mathbf{a}}_3^r)$, such that $\vec{\mathbf{a}}_3^r = \vec{\mathbf{a}}_3$.

Working backwards, we see that the purpose of the second rotation is to rotate $\vec{\mathbf{a}}_3^q$ about $\vec{\mathbf{a}}_2^q$ to $\vec{\mathbf{a}}_3^r = \vec{\mathbf{a}}_3$. Since $\vec{\mathbf{a}}_2^q \perp \vec{\mathbf{a}}_3^q$, this rotation is possible only if $\vec{\mathbf{a}}_2^q \perp \vec{\mathbf{a}}_3$ (which puts both $\vec{\mathbf{a}}_3^q$ and $\vec{\mathbf{a}}_3$ in the plane perpendicular to $\vec{\mathbf{a}}_2^q$). Therefore, the purpose of the first rotation is to rotate $\vec{\mathbf{a}}_2^a$ about $\vec{\mathbf{a}}_1^a$ such that $\vec{\mathbf{a}}_2^q \perp \vec{\mathbf{a}}_3$.

We shall now obtain such a rotation sequence, with the angles in the required ranges as in Theorem 3.

Step 1: First Rotation

The objective of the first rotation is to rotate $\vec{\mathbf{a}}_2^a$ about $\vec{\mathbf{a}}_1^a$ such that the resulting $\vec{\mathbf{a}}_2^q$ is perpendicular to $\vec{\mathbf{a}}_3$. Let $\mathcal{P}_{a_1}$ and $\mathcal{P}_{a_3}$ denote the planes perpendicular to $\vec{\mathbf{a}}_1 = \vec{\mathbf{a}}_1^a$ and $\vec{\mathbf{a}}_3$ respectively. Then, $\vec{\mathbf{a}}_2^q$ must lie along the line of intersection between these two planes, as shown in Figure 1. If $\vec{\mathbf{a}}_1 = \pm\vec{\mathbf{a}}_3$, then these planes are identical, and the first rotation angle can take any value from $\phi_1 \in (-\pi, \pi]$. As we shall demonstrate later, this corresponds to the singularity of the generalized Euler sequence. If $\vec{\mathbf{a}}_a \neq \pm\vec{\mathbf{a}}_3$, then the line of intersection between the planes yields two possible rotation angles $\bar{\phi}_1$ and ϕ_1'

between in the range $\phi_1 \in (-\pi, \pi]$, separated by π. As is evident from Figure 1, the two rotations yield mirror images of $\overrightarrow{\mathbf{a}}_3$ about $\overrightarrow{\mathbf{a}}_1^q$, as seen in a frame fixed to $\overrightarrow{\mathbf{a}}_1^q$ and $\overrightarrow{\mathbf{a}}_2^q$. This is further shown in Figure 2.

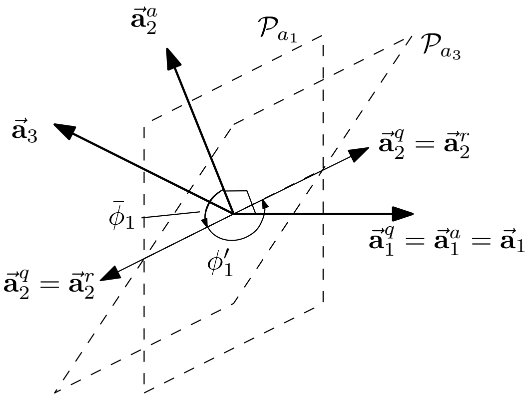

Fig. 1 First rotation

Step 2: Second Rotation
The objective of the second rotation is to rotate $\overrightarrow{\mathbf{a}}_3^q$ about $\overrightarrow{\mathbf{a}}_2^q$ such that the resulting $\overrightarrow{\mathbf{a}}_3^r = \overrightarrow{\mathbf{a}}_3$. Let $\alpha \in (-\pi, \pi]$ be the angle $\overrightarrow{\mathbf{a}}_3^q$ makes with $\overrightarrow{\mathbf{a}}_1^q$, measured in the left-hand sense about $\overrightarrow{\mathbf{a}}_2^q$, as shown in Figure 2. Then, as shown in Figure 2, since the two possible rotations in Step 1 yield mirror images of $\overrightarrow{\mathbf{a}}_3$ about $\overrightarrow{\mathbf{a}}_1^q$, as seen in a frame fixed to $\overrightarrow{\mathbf{a}}_1^q$ and $\overrightarrow{\mathbf{a}}_2^q$, it is always possible to find a ϕ_1 from Step 1, such that $\overrightarrow{\mathbf{a}}_3$ makes angle $\phi_2 - \alpha \in [0, \pi]$ from $\overrightarrow{\mathbf{a}}_1^q$, measured in the right-hand sense about $\overrightarrow{\mathbf{a}}_2^q$. Likewise, it is always possible to find a ϕ_1 from Step 1, such that $\overrightarrow{\mathbf{a}}_3$ makes angle $\phi_2 - \alpha \in [-\pi, 0]$ from $\overrightarrow{\mathbf{a}}_1^q$, measured in the right-hand sense about $\overrightarrow{\mathbf{a}}_2^q$. The corresponding angle ϕ_2 is precisely the angle of rotation sought in Step 2, which rotates $\overrightarrow{\mathbf{a}}_3^q$ about $\overrightarrow{\mathbf{a}}_2^q$ such that $\overrightarrow{\mathbf{a}}_3^r = \overrightarrow{\mathbf{a}}_3$.
Step 3: Third Rotation
As already noted, since the first two rotations resulted in $\overrightarrow{\mathbf{a}}_3^r = \overrightarrow{\mathbf{a}}_3$, it follows that an angle $\phi_3 \in (-\pi, \pi]$ can be found such that when $\mathcal{F}_r$ is rotated about $\overrightarrow{\mathbf{a}}_3^r = \overrightarrow{\mathbf{a}}_3$, it results in $\mathcal{F}_b$.

This geometric description of the generalized Euler rotation sequence completely parallels the algebraic proof of Theorem 3.

Physical Interpretation of the Singularity

We shall now provide a physical interpretation of the singularity condition given in (40). Associated with a generalized Euler sequence, consider again the frames $\mathcal{F}_a$ and $\mathcal{F}_b$ and the axes $\overrightarrow{\mathbf{a}}_1 = \overrightarrow{\mathcal{F}}_a^T \mathbf{a}_1$ and $\overrightarrow{\mathbf{a}}_3 = \overrightarrow{\mathcal{F}}_b^T \mathbf{a}_3$ from the

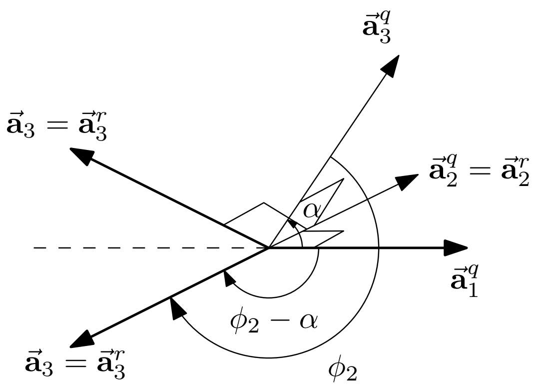

Fig. 2 Second rotation

previous subsection. Then, the matrix $\mathbf{C} = \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{C}_{a_1}(\phi_1)$ transforms coordinates from frame $\mathcal{F}_a$ to $\mathcal{F}_b$. As such, $\underrightarrow{\mathbf{a}}_1 = \underrightarrow{\mathcal{F}}_a^T\mathbf{a}_1 = \underrightarrow{\mathcal{F}}_b^T\mathbf{C}\mathbf{a}_1$. Consequently, the singularity condition in (40) becomes

$$\underrightarrow{\mathbf{a}}_1 = \pm\underrightarrow{\mathbf{a}}_3. \tag{43}$$

That is, the singularity associated with the rotation sequence occurs when the first and third rotations occur about the same physical axis, showing that the physical meaning of the singularity for standard Euler sequences extends directly to generalized Euler sequences. A physical explanation of the singularity is readily seen in Figure 1. Specifically, at the singularity, the planes $\mathcal{P}_{a_1}$ and $\mathcal{P}_{a_3}$ coincide. As already explained, this results in a continuum of solutions for ϕ_1. Correspondingly, there is a continuum of solutions for ϕ_3.

Kinematics of the Generalized Euler Sequence

For completeness, we also examine the kinematics of the generalized Euler sequence.

Suppose that $\mathbf{C}(t) : \mathbb{R} \to SO(3)$ is continuously differentiable. Then, there exists a continuous $\boldsymbol{\omega}(t) : \mathbb{R} \to \mathbb{R}^3$, such that [4]

$$\dot{\mathbf{C}} = -\boldsymbol{\omega}^\times\mathbf{C}. \tag{44}$$

According to Theorem 3, any $\mathbf{C} \in SO(3)$ can be written as a generalized Euler sequence as given in (30). Then, as shown in [5], the rotation matrix kinematics in (44) is equivalent to the generalized Euler angle kinematics given by

$$\boldsymbol{\omega} = \mathbf{S}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}, \tag{45}$$

where $\boldsymbol{\phi} = [\phi_1 \;\; \phi_2 \;\; \phi_3]^T$, and

$$\mathbf{S}(\boldsymbol{\phi}) = \left[\, \mathbf{C}_{a_3}(\phi_3)\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1 \;\; \mathbf{C}_{a_3}(\phi_3)\mathbf{a}_2 \;\; \mathbf{a}_3 \,\right]. \tag{46}$$

Now, $\mathbf{S}(\boldsymbol{\phi}) \in \mathbb{R}^{3\times3}$, and is invertible if and only if $\mathbf{S}(\boldsymbol{\phi})^T\mathbf{S}(\boldsymbol{\phi})$ is invertible. Furthermore, if it is invertible, then

$$\mathbf{S}(\boldsymbol{\phi})^{-1} = (\mathbf{S}(\boldsymbol{\phi})^T\mathbf{S}(\boldsymbol{\phi}))^{-1}\mathbf{S}(\boldsymbol{\phi})^T, \tag{47}$$

which is readily verified by direct multiplication. Denoting for conciseness $\mathbf{C}_1 = \mathbf{C}_{a_1}(\phi_1)$, $\mathbf{C}_2 = \mathbf{C}_{a_2}(\phi_2)$, and $\mathbf{C}_3 = \mathbf{C}_{a_3}(\phi_3)$, by direct multiplication and application of Lemma 4 part 1, we obtain

$$\begin{aligned}\mathbf{S}(\boldsymbol{\phi})^T\mathbf{S}(\boldsymbol{\phi}) &= \begin{bmatrix} \mathbf{a}_1^T\mathbf{C}_2^T\mathbf{C}_3^T\mathbf{C}_3\mathbf{C}_2\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{C}_2^T\mathbf{C}_3^T\mathbf{C}_3\mathbf{a}_2 & \mathbf{a}_1^T\mathbf{C}_2^T\mathbf{C}_3^T\mathbf{a}_3 \\ \mathbf{a}_2^T\mathbf{C}_3^T\mathbf{C}_3\mathbf{C}_2\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{C}_3^T\mathbf{C}_3\mathbf{a}_2 & \mathbf{a}_2^T\mathbf{C}_3^T\mathbf{a}_3 \\ \mathbf{a}_3^T\mathbf{C}_3\mathbf{C}_2\mathbf{a}_1 & \mathbf{a}_3^T\mathbf{C}_3\mathbf{a}_2 & \mathbf{a}_3^T\mathbf{a}_3 \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 & \mathbf{a}_1^T\mathbf{C}_2^T\mathbf{a}_3 \\ 0 & 1 & 0 \\ \mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1 & 0 & 1 \end{bmatrix},\end{aligned} \tag{48}$$

from which we obtain

$$\det[\mathbf{S}(\boldsymbol{\phi})^T\mathbf{S}(\boldsymbol{\phi})] = 1 - (\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1)^2. \tag{49}$$

This shows that $\mathbf{S}(\boldsymbol{\phi})$ is singular if and only if $\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1 = \pm1$, which is precisely the singularity condition in (37).

Finally, assuming $\mathbf{a}_3^T\mathbf{C}_{a_2}(\phi_2)\mathbf{a}_1 \neq \pm1$, we obtain from (48) and (49)

$$(\mathbf{S}(\boldsymbol{\phi})^T\mathbf{S}(\boldsymbol{\phi}))^{-1} = \begin{bmatrix} \frac{1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} & 0 & \frac{-\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} \\ 0 & 1 & 0 \\ \frac{-\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} & 0 & \frac{1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} \end{bmatrix}, \tag{50}$$

Substitution of (46) and (50) into (47) gives

$$\begin{aligned}\mathbf{S}(\boldsymbol{\phi})^{-1} &= (\mathbf{S}(\boldsymbol{\phi})^T\mathbf{S}(\boldsymbol{\phi}))^{-1}\mathbf{S}(\boldsymbol{\phi})^T, \\ &= \begin{bmatrix} \frac{1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} & 0 & \frac{-\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} \\ 0 & 1 & 0 \\ \frac{-\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} & 0 & \frac{1}{1-(\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2} \end{bmatrix} \begin{bmatrix} (\mathbf{C}_3\mathbf{C}_2\mathbf{a}_1)^T \\ (\mathbf{C}_3\mathbf{a}_2)^T \\ \mathbf{a}_3^T \end{bmatrix},\end{aligned}$$

which leads to

$$\mathbf{S}(\boldsymbol{\phi})^{-1} = \begin{bmatrix} \left(\mathbf{C}_3\mathbf{C}_2\mathbf{a}_1 - (\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)\mathbf{a}_3\right)^T / \left(1 - (\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2\right) \\ (\mathbf{C}_3\mathbf{a}_2)^T \\ \left(\mathbf{a}_3 - (\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)\mathbf{C}_3\mathbf{C}_2\mathbf{a}_1\right)^T / \left(1 - (\mathbf{a}_3^T\mathbf{C}_2\mathbf{a}_1)^2\right) \end{bmatrix}. \tag{51}$$

Consequently, we see that the generalized Euler kinematics in (45) are invertible, if and only if the generalized Euler sequence is not at its singularity.

Concluding Remarks

We have presented a comprehensive, self-contained, treatment of the generalized Euler sequences. In particular, we have taken a constructive approach to proving that the generalized Euler sequences provide a universal representation of $SO(3)$. In doing so, we have obtained specific ranges that contain the associated generalized Euler angles. We have characterized the singularity condition of the generalized Euler sequences, and shown that the generalized Euler angles are uniquely specified within their restricted ranges when away from the singularity. A physical interpretation of the generalized Euler sequences and the associated singularity has been provided. For completeness, a treatment of the generalized Euler kinematics has been included also.

References

1. Bernstein, D.S.: Matrix Mathematics, 2nd edn. Princeton University Press, Princeton, New Jersey (2009)
2. Davenport, P.D.: Rotations about Nonorthogonal Axes. AIAA Journal **11**(6), 853–857 (1973)
3. Euler, L.: De Motu Corporum Circa Punctum Fixum Mobilium. Commentatio 825 indicis ENESTROEMIANI, Opera posthuma **2**, 43–62 (1862)
4. Hughes, P.C.: Spacecraft Attitude Dynamics, 2nd edn. Dover, Mineola, New York (2004)
5. Markley, F.L., Crassidis, J.L.: Fundamentals of Spacecraft Attitude Determination and Control. Springer, New York (2014)
6. Shuster, M.D.: A Survey of Attitude Representations. Journal of the Astronautical Sciences **41**(4), 439–517 (1993)
7. Shuster, M.D., Markley, F.L.: General Formula for Extracting the Euler Angles. AIAA Journal of Guidance, Control, and Dynamics **29**(1), 215–217 (2003)
8. Shuster, M.D., Markley, F.L.: Generalization of the Euler Angles. Journal of the Astronautical Sciences **51**(2), 123–132 (2003)
9. Wittenburg, J., Lilov, L.: Decomposition of a Finite Rotation into Three Rotations about Given Axes. Multibody System Dynamics **9**, 353–375 (2003)