Computationally Simple Sub-Optimal Filtering for Spacecraft Motion Estimation

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Abstract

This paper presents a computationally simple near-optimal filter for spacecraft motion estimation. This is particularly important in applications where the computational resources are very limited, such as in cube-satellite and nano-satellite missions. The proposed filter consists of two scalar gains, and has analytically guaranteed performance under given bounds on the process and measurement noise covariances. Unlike the Kalman filter or its variants, there is no associated covariance propagation. Favorable performance of the presented filter, compared with a conventional extended Kalman filter, is demonstrated via a hardware-in-the-loop simulation of a dual spacecraft formation navigation problem.

Key words: Space vehicles, Navigation, Kalman filters, Extended Kalman filters, Computational Simplicity

1 Introduction

There are increasing efforts to make spacecraft more autonomous in general. An enabling technology for this is the ability to estimate the spacecraft motion onboard in real-time. In particular, much of the current effort is focused on absolute and relative orbital motion estimation for spacecraft formations (Gill et al. (2007), Eyer and Damaren (2009), D'Amico and Montenbruck (2010), Ardaens et al. (2013), Ebinuma et al. (2002, 2003)). The benchmark algorithm for on-board motion estimation is the Kalman filter and its variants (Crassidis and Junkins (2012), El-Sheimy et al. (2006)).

A common present day assumption is that computational power is no longer a limitation. Consequently, there is a general trend toward increasing complexity of on-board state estimation algorithms, with the goal of improving both estimation robustness and performance. However, there is still real value in developing and utilizing computationally simple algorithms when the spacecraft performance requirements allow for it. First, a number of past space-system failures can be attributed to software errors (Harland and Lorenz (2005), Tafazoli (2009)). With the ever increasing complexity of flight software, these failures are becoming more difficult to predict, detect, isolate and mitigate. Second, simple algorithms are more amenable to analysis with the ability to make conclusions based on rigorous mathematical arguments, while this is more difficult or impossible to do with more complex algorithms. For example, with some highly complex implementations of the extended Kalman filter (D'Amico and Montenbruck (2010)) or its variants such as the unscented Kalman filter (Wolfe et al. (2007), Crassidis and Junkins (2012)), there are no analytical guarantees of filter stability or consistency. As such, engineers must embark on extensive numerical simulation campaigns in order to assess filter convergence and steady-state performance. Finally, in very small satellite applications (for example, cube-satellites and nano-satellites), the onboard processors that are used are still typically quite primitive, with limited computational resources.

In this paper, a novel computationally simple continuous-discrete filter is developed for spacecraft orbital motion estimation based on position measurements. The filter contains two scalar gains, and requires no covariance propagation. Each gain is separated into two parts: a time-varying transient part, and a constant steady-state part. The transient part of the gains are applied for a pre-determined period of time, with the objective of achieving rapid filter convergence similar to the Kalman filter. After the transient period, the gains are switched to the constant steady-state part, and they provide the long-term filter stability and performance. The filter is

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similar to the Kalman filter in that it contains a linear correction term based on the measurement innovation. However, unlike the Kalman filter, the correction is applied directly inside the state estimate propagation step, rather than after the propagation step, resulting in an a-priori filter (a one-step ahead predictor). This key difference allows the filter's stability and steady-state performance to be analyzed, and analytically guaranteed steady-state performance bounds are obtained. The effectiveness of the proposed filter is demonstrated by applying it to a hardware-in-the-loop simulation for GPS-based relative navigation for a close spacecraft formation.

2 Problem Formulation

In this paper, the $n \times n$ identity matrix will be denoted by $\mathbf{I}_n$, the $n \times n$ matrix of zeros by $\mathbf{0}_n$, and $\|\mathbf{x}\|$ denotes the Euclidean 2-norm of the vector $\mathbf{x} \in \mathbb{R}^n$, while $\|\mathbf{X}\|$ denotes the induced 2-norm of the matrix $\mathbf{X} \in \mathbb{R}^{n\times m}$.

It is assumed that position measurements are available (for example, from a GPS receiver), of the form

$$\mathbf{r}^m(t_k) = \mathbf{r}(t_k) + \mathbf{n}_k, \tag{1}$$

expressed in some reference frame of interest $\mathcal{F}_x$ (common choices are inertial or Earth-fixed frames). The vectors $\mathbf{r}^m(t_k)$ and $\mathbf{r}(t_k)$ represent respectively the measured and true spacecraft position vectors at time instant t_k, and $\mathbf{n}_k$ is a zero-mean white noise sequence with covariance $E\{\mathbf{n}_k\mathbf{n}_j^T\} = \mathbf{R}_k\delta_{kj}$, where $E\{\cdot\}$ denotes the expectation operator, δ_{kj} is the discrete delta function and $\mathbf{R}_k$ is a symmetric positive definite matrix.

In $\mathcal{F}_x$ coordinates, the spacecraft translational equations of motion are given by

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \mathbf{v}(t) - \boldsymbol{\omega}_x^\times(t)\mathbf{r}(t),\\ \dot{\mathbf{v}}(t) &= \mathbf{a}_g(\mathbf{r}(t),t) - \boldsymbol{\omega}_x^\times(t)\mathbf{v}(t) + \mathbf{a}_{ng}(t) + \mathbf{w}_a(t),\end{aligned} \tag{2}$$

where $\mathbf{v}(t)$ is the spacecraft inertial velocity, $\boldsymbol{\omega}_x(t)$ is the known inertial angular velocity of $\mathcal{F}_x$, $\mathbf{a}_g(\mathbf{r}(t),t)$ is the spacecraft gravitational acceleration, $\mathbf{a}_{ng}(t)$ is the modeled spacecraft non-gravitational acceleration, $\mathbf{w}_a(t)$ represents un-modeled accelerations and modelling errors. For any $\mathbf{a} \in \mathbb{R}^3$, the matrix $\mathbf{a}^\times \in \mathbb{R}^{3\times 3}$ is the skew symmetric matrix such that $\mathbf{a}^\times\mathbf{b}$ gives the cross-product between $\mathbf{a} \in \mathbb{R}^3$ and any other $\mathbf{b} \in \mathbb{R}^3$. It is assumed that $\mathbf{a}_g(\mathbf{r},t)$ is continuously differentiable in $\mathbf{r}$. Furthermore, the gravitational acceleration is derivable from a potential function $\phi_g(\mathbf{r},t)$ as $\mathbf{a}_g(\mathbf{r},t)^T = \nabla\phi_g(\mathbf{r},t)$ Vallado (2004). Consequently, the Jacobian $\partial\mathbf{a}_g(\mathbf{r},t)/\partial\mathbf{r}$ is symmetric. It is assumed that $\mathbf{w}_a(t)$ is a zero-mean white noise process with covariance $E\{\mathbf{w}_a(t)\mathbf{w}_a(\tau)^T\} = \bar{\mathbf{Q}}_t\delta(t-\tau)$ where $\bar{\mathbf{Q}}(t)$ is a symmetric positive semi-definite matrix, and $\delta(t-\tau)$ is the Dirac delta function.

The following filter structure is now imposed

$$\begin{aligned}\dot{\hat{\mathbf{r}}}(t) &= \hat{\mathbf{v}}(t) - \boldsymbol{\omega}_x^\times(t)\hat{\mathbf{r}}(t) + \mathbf{u}_r(t),\\ \dot{\hat{\mathbf{v}}}(t) &= \mathbf{a}_g(\hat{\mathbf{r}}(t),t) - \boldsymbol{\omega}_x^\times(t)\hat{\mathbf{v}}(t) + \mathbf{a}_{ng}(t) + \mathbf{u}_v(t),\end{aligned} \tag{3}$$

where $\hat{\mathbf{r}}$ and $\hat{\mathbf{v}}$ are the estimates of $\mathbf{r}$ and $\mathbf{v}$, respectively, and $\mathbf{u}_r$ and $\mathbf{u}_v$ are control-like inputs, which are yet to be defined. The position and velocity estimation errors are defined as $\tilde{\mathbf{r}}(t) = \mathbf{r}(t) - \hat{\mathbf{r}}(t)$ and $\tilde{\mathbf{v}}(t) = \mathbf{v}(t) - \hat{\mathbf{v}}(t)$ respectively. Using equation (1), the difference between the measured and estimated position becomes

$$\mathbf{r}^m(t_k) - \hat{\mathbf{r}}(t_k) = \tilde{\mathbf{r}}(t_k) + \mathbf{n}_k, \tag{4}$$

which may be used as an input for the filter in (3). It is assumed that $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{v}}$ are small, such that their dynamics are well described by linearizing their dynamics about the estimates, $\hat{\mathbf{r}}(t)$ and $\hat{\mathbf{v}}(t)$. Using equations (2) and (3), the linearized error dynamics take the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \tilde{\mathbf{A}}(t)\mathbf{x}(t) - \mathbf{u}(t) + \mathbf{B}\mathbf{w}_a(t), \tag{5}$$

together with a discrete-time measurement equation from (4) of the form

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}(t_k) + \mathbf{n}_k, \tag{6}$$

where

$$\mathbf{x} \triangleq \begin{bmatrix}\tilde{\mathbf{r}}\\ \tilde{\mathbf{v}}\end{bmatrix},\ \ \mathbf{y}_k = \mathbf{r}^m(t_k) - \hat{\mathbf{r}}(t_k),\ \ \mathbf{A} = \begin{bmatrix}\mathbf{0}_3 & \mathbf{I}_3\\ \mathbf{0}_3 & \mathbf{0}_3\end{bmatrix},$$

$$\tilde{\mathbf{A}}(t) = \begin{bmatrix}-\boldsymbol{\omega}_x^\times(t) & \mathbf{0}_3\\ \partial\mathbf{a}_g(t) & -\boldsymbol{\omega}_x^\times(t)\end{bmatrix},\ \ \partial\mathbf{a}_g(t) \triangleq \partial\mathbf{a}_g(\hat{\mathbf{r}}(t),t)/\partial\mathbf{r},$$

$$\mathbf{B} = \begin{bmatrix}\mathbf{0}_3 & \mathbf{I}_3\end{bmatrix}^T,\ \ \mathbf{u} = \begin{bmatrix}\mathbf{u}_r^T & \mathbf{u}_v^T\end{bmatrix}^T,\ \ \mathbf{H} = \begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3\end{bmatrix}.$$

From this point on, for a time-dependent vector or matrix $\mathbf{X}(t)$, we will denote $\mathbf{X}(t_k) = \mathbf{X}_k$. It is assumed that the sample period $T = t_{k+1} - t_k$ is short enough such that the linearized error dynamics in (5) can be discretized as

$$\mathbf{x}_{k+1} = e^{(\mathbf{A}+\tilde{\mathbf{A}}_k)T}\mathbf{x}_k - \int_{t_k}^{t_{k+1}} e^{(\mathbf{A}+\tilde{\mathbf{A}}_k)(t_{k+1}-\tau)}\mathbf{u}(\tau)d\tau + \mathbf{w}_k, \tag{7}$$

where $\mathbf{w}_k$ is a zero-mean white noise sequence with covariance

$$\mathbf{Q}_k = \int_{t_k}^{t_{k+1}} e^{(\mathbf{A}+\tilde{\mathbf{A}}_k)(t_{k+1}-\tau)}\mathbf{B}\bar{\mathbf{Q}}_\tau\mathbf{B}^T e^{(\mathbf{A}+\tilde{\mathbf{A}}_k)^T(t_{k+1}-\tau)}d\tau. \tag{8}$$

In typical problems involving spacecraft, $\tilde{\mathbf{A}}_k$ is small compared with $\mathbf{A}$, and it may be viewed as a perturbation to $\mathbf{A}$. One readily finds that

$$e^{\mathbf{A}t} = \begin{bmatrix} \mathbf{I}_3 & t\mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{9}$$

Using (9), if $\bar{\mathbf{Q}}_t = q(t)\mathbf{I}_3$ (i.e. isotropic) where $q(t) \geq 0$, then in the unperturbed case (i.e. $\tilde{\mathbf{A}}_k = \mathbf{0}$) the discrete process noise in (8) takes the form

$$\mathbf{Q}_k = \int_{t_k}^{t_{k+1}} \begin{bmatrix} (t_{k+1}-\tau)^2\mathbf{I}_3 & (t_{k+1}-\tau)\mathbf{I}_3 \\ (t_{k+1}-\tau)\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} q(\tau)d\tau. \tag{10}$$

Considering the matrix exponential expansion for $\exp[(\mathbf{A}+\tilde{\mathbf{A}}_k)t]$, and keeping only first order terms in $\tilde{\mathbf{A}}_k$ gives

$$e^{(\mathbf{A}+\tilde{\mathbf{A}}(t_k))t} \approx e^{\mathbf{A}t} + \begin{bmatrix} \frac{\partial \mathbf{a}_{g,k} t^2}{2} & \frac{\partial \mathbf{a}_{g,k} t^3}{6} \\ \partial \mathbf{a}_{g,k} t & \frac{\partial \mathbf{a}_{g,k} t^2}{2} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\omega}_{x,k}^{\times} t & \boldsymbol{\omega}_{x,k}^{\times} t^2 \\ \mathbf{0}_3 & \boldsymbol{\omega}_{x,k}^{\times} t \end{bmatrix}. \tag{11}$$

The control term in (7) will now be examined. Since the measurements in (6) are available only at the sample times, the control-like input in equations (3) and (5) is implemented through a zero-order hold

$$\mathbf{u}(t) = \bar{\mathbf{u}}_k, \;\; t_k \leq t < t_{k+1}, \tag{12}$$

for some yet to be chosen $\bar{\mathbf{u}}_k$. Temporarily neglecting the perturbing terms in (11), one obtains

$$\int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{u}(\tau)d\tau = \begin{bmatrix} T\mathbf{I}_3 & (T^2/2)\mathbf{I}_3 \\ \mathbf{0}_3 & T\mathbf{I}_3 \end{bmatrix} \bar{\mathbf{u}}_k =: \breve{\mathbf{u}}_k \tag{13}$$

Using equations (9) and (13), the ideal discrete system corresponding to (7) becomes

$$\mathbf{x}_{k+1} = \boldsymbol{\Phi}\mathbf{x}_k - \breve{\mathbf{u}}_k + \mathbf{w}_k, \tag{14}$$

where $\boldsymbol{\Phi} \stackrel{\Delta}{=} \exp[\mathbf{A}T]$, and $\breve{\mathbf{u}}_k$ is a discrete control-like input, which is yet to be defined. Equation (13) is readily inverted to obtain

$$\bar{\mathbf{u}}_k = \begin{bmatrix} (1/T)\mathbf{I}_3 & -(1/2)\mathbf{I}_3 \\ \mathbf{0}_3 & (1/T)\mathbf{I}_3 \end{bmatrix} \breve{\mathbf{u}}_k, \tag{15}$$

which can then be implemented continuously through the zero-order hold in (12).

Re-introducing the perturbing terms in (11), and using (15), the discretization in (7) can now be written as

$$\mathbf{x}_{k+1} = (\boldsymbol{\Phi} + \tilde{\boldsymbol{\Phi}}_k)\mathbf{x}_k - (\mathbf{I}_6 + \tilde{\mathbf{B}}_k)\breve{\mathbf{u}}_k + \mathbf{w}_k, \tag{16}$$

where

$$\tilde{\boldsymbol{\Phi}}_k = \begin{bmatrix} \frac{\partial \mathbf{a}_{g,k} T^2}{2} & \frac{\partial \mathbf{a}_{g,k} T^3}{6} \\ \partial \mathbf{a}_{g,k} T & \frac{\partial \mathbf{a}_{g,k} T^2}{2} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\omega}_{x,k}^{\times} T & \boldsymbol{\omega}_{x,k}^{\times} T^2 \\ \mathbf{0}_3 & \boldsymbol{\omega}_{x,k}^{\times} T \end{bmatrix}, \tag{17}$$

and

$$\tilde{\mathbf{B}}_k = \begin{bmatrix} \frac{\partial \mathbf{a}_{g,k} T^2}{6} & -\frac{\partial \mathbf{a}_{g,k} T^3}{24} \\ \frac{\partial \mathbf{a}_{g,k} T}{2} & -\frac{\partial \mathbf{a}_{g,k} T^2}{12} \end{bmatrix} - \begin{bmatrix} \frac{\boldsymbol{\omega}_{x,k}^{\times} T}{2} & \frac{\boldsymbol{\omega}_{x,k}^{\times} T^2}{12} \\ \mathbf{0}_3 & \frac{\boldsymbol{\omega}_{x,k}^{\times} T}{2} \end{bmatrix}. \tag{18}$$

Finally, partitioning $\breve{\mathbf{u}}_k = \begin{bmatrix} \breve{\mathbf{u}}_{r,k}^T & \breve{\mathbf{u}}_{v,k}^T \end{bmatrix}^T$ and $\mathbf{w}_k = \begin{bmatrix} \mathbf{w}_{r,k}^T & \mathbf{w}_{v,k}^T \end{bmatrix}^T$, where $\breve{\mathbf{u}}_{r,k}, \breve{\mathbf{u}}_{v,k}, \mathbf{w}_{r,k}, \mathbf{w}_{v,k} \in \mathbb{R}^3$, and using the unitary transformation

$$\mathbf{z} = \mathbf{T}\mathbf{x}, \tag{19}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_4 & \mathbf{e}_2 & \mathbf{e}_5 & \mathbf{e}_3 & \mathbf{e}_6 \end{bmatrix}^T, \tag{20}$$

and $\mathbf{e}_1, ..., \mathbf{e}_6 \in \mathbb{R}^6$ denotes the standard basis for $\mathbb{R}^6$, the ideal discrete system in (14), together with the measurement equation in (6), can be decoupled into three identical equations, given by

$$\begin{aligned} \mathbf{z}_{k+1,i} &= \hat{\boldsymbol{\Phi}}\mathbf{z}_{k,i} - \breve{\mathbf{u}}_{k,i} + \mathbf{w}_{k,i}, \\ y_{k,i} &= \hat{\mathbf{H}}\mathbf{z}_{k,i} + n_{k,i}, \end{aligned} \tag{21}$$

for $i = 1, 2, 3$, where

$$\mathbf{z}_{k,i} = \begin{bmatrix} \tilde{r}_{k,i} & \tilde{v}_{k,i} \end{bmatrix}^T, \;\; \breve{\mathbf{u}}_{k,i} = \begin{bmatrix} \breve{u}_{r,k,i} & \breve{u}_{v,k,i} \end{bmatrix}^T,$$

$$\mathbf{w}_{k,i} = \begin{bmatrix} w_{r,k,i} & w_{v,k,i} \end{bmatrix}^T, \;\; \hat{\boldsymbol{\Phi}} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \;\; \hat{\mathbf{H}} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{22}$$

and $\tilde{r}_{k,i}$, $\tilde{v}_{k,i}$, $\breve{u}_{r,k,i}$, $\breve{u}_{v,k,i}$, $w_{r,k,i}$, $w_{v,k,i}$, $y_{k,i}$, and $n_{k,i}$ denote the i^{th} component of $\tilde{\mathbf{r}}_k$, $\tilde{\mathbf{v}}_k$, $\mathbf{u}_{r,k}$, $\mathbf{u}_{k,v}$, $\mathbf{w}_{k,r}$, $\mathbf{w}_{k,v}$, $\mathbf{y}_k$, and $\mathbf{n}_k$, respectively.

3 Development of A-Priori Filter

In view of the error equations in (16) together with the measurement equation in (6), consider a discrete-time system given by

$$\begin{aligned} \mathbf{x}_{k+1} &= \boldsymbol{\Phi}_k\mathbf{x}_k + \mathbf{w}_k - \breve{\mathbf{u}}_k, \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{n}_k, \end{aligned} \tag{23}$$

where $\{\mathbf{w}_k\}$ and $\{\mathbf{n}_k\}$ are uncorrelated zero-mean white noise sequences with covariances $E\{\mathbf{w}_k\mathbf{w}_j^T\} = \mathbf{Q}_k\delta_{kj}$ and $E\{\mathbf{n}_k\mathbf{n}_j^T\} = \mathbf{R}_k\delta_{kj}$, where $\mathbf{Q}_k$ is a symmetric positive semi-definite matrix, and $\mathbf{R}_k$ is a symmetric positive-definite matrix. It is further assumed that $\mathbf{x}_0$ is a zero-mean random variable that is uncorrelated with the sequences $\{\mathbf{w}_k\}$ and $\{\mathbf{n}_k\}$, with covariance $E\{\mathbf{x}_0\mathbf{x}_0^T\} = \mathbf{P}_0$, where $\mathbf{P}_0$ is symmetric and positive definite. The control-like input in (23) is chosen to have the form

$$\breve{\mathbf{u}}_k = \mathbf{K}_k\mathbf{y}_k, \tag{24}$$

and substitution of (24) into (23), gives

$$\mathbf{x}_{k+1} = (\boldsymbol{\Phi}_k - \mathbf{K}_k\mathbf{H}_k)\mathbf{x}_k + \mathbf{w}_k - \mathbf{K}_k\mathbf{n}_k. \tag{25}$$

Under the assumed conditions, $\{\mathbf{x}_k\}$ is a zero-mean random sequence, with covariance $\mathbf{P}_k = E\{\mathbf{x}_k\mathbf{x}_k^T\}$ satisfying the recursion

$$\mathbf{P}_{k+1} = (\boldsymbol{\Phi}_k - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k(\boldsymbol{\Phi}_k - \mathbf{K}_k\mathbf{H}_k)^T + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^T + \mathbf{Q}_k. \tag{26}$$

The control gain $\mathbf{K}_k$ is then chosen as in the Kalman filter to minimize $trace[\mathbf{P}_{k+1}]$ Crassidis and Junkins (2012), Chui and Chen (1999), which leads to

$$\mathbf{K}_k = \boldsymbol{\Phi}_k\mathbf{P}_k\mathbf{H}_k^T\bar{\mathbf{R}}_k^{-1}, \tag{27}$$

where $\bar{\mathbf{R}}_k = \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^T + \mathbf{R}_k$. Substitution of (27) into (26) leads to the Riccati recursion for the covariance

$$\mathbf{P}_{k+1} = \boldsymbol{\Phi}_k\mathbf{P}_k\boldsymbol{\Phi}_k^T - \boldsymbol{\Phi}_k\mathbf{P}_k\mathbf{H}_k^T\bar{\mathbf{R}}_k^{-1}\mathbf{H}_k\mathbf{P}_k\boldsymbol{\Phi}_k^T + \mathbf{Q}_k. \tag{28}$$

The recursions in (25) and (28) are identical in form to the recursions for the a-priori form of the Kalman filter Crassidis and Junkins (2012).

Using the above development as the basis, a discrete-time sub-optimal filter is presented next. This filter is similar in spirit to the continuous-time filters presented in Kim (1990), de Ruiter (2010, 2012). The sub-optimal filter is based on the ideal discrete system in equation (14) with measurements in (6). As such, in equations (23) to (28), set

$$\boldsymbol{\Phi}_k = \boldsymbol{\Phi}, \;\; \mathbf{B}_k = \mathbf{B}, \;\; \mathbf{H}_k = \mathbf{H}. \tag{29}$$

Furthermore, it is assumed that it is possible to find $q_{rr} > 0$, $q_{vv} > 0$, q_{rv}, and $r > 0$ such that

$$r\mathbf{I}_3 \geq \mathbf{R}_k, \text{ and } \mathbf{Q} \triangleq \begin{bmatrix} q_{rr}\mathbf{I}_3 & q_{rv}\mathbf{I}_3 \\ q_{rv}\mathbf{I}_3 & q_{vv}\mathbf{I}_3 \end{bmatrix} \geq \mathbf{Q}_k, \tag{30}$$

for all $k \geq 0$ where $\mathbf{R}_k$ and $\mathbf{Q}_k$ are the true measurement and process noise covariances corresponding to equations (1) and (7). The form of $\mathbf{Q}$ in (30) is motivated by the ideal case in (10). For the purposes of filter development, it will be assumed that

$$\mathbf{R}_k = r\mathbf{I}_3, \;\; \mathbf{Q}_k = \mathbf{Q}, \;\; \forall k \geq 0. \tag{31}$$

The matrices in (29) and (31) add some conservatism, but simplify the filter design. Inspired by Kim (1990), de Ruiter (2010, 2012), simple analytical approximations are sought to the solution of (28), and consequently to the gain $\mathbf{K}_k$.

It is readily checked that $(\boldsymbol{\Phi}, \mathbf{H})$ is observable, and $(\boldsymbol{\Phi}, \mathbf{Q}^{1/2})$ is controllable. Therefore, the covariance $\mathbf{P}_k$ in (28) with matrices given in (29) and (31) converges to a positive definite steady-state value $\mathbf{P}_{ss}$, satisfying (Chui and Chen (1999))

$$\mathbf{P}_{ss} = \boldsymbol{\Phi}\mathbf{P}_{ss}\boldsymbol{\Phi}^T - \boldsymbol{\Phi}\mathbf{P}_{ss}\mathbf{H}^T\bar{\mathbf{R}}^{-1}\mathbf{H}\mathbf{P}_{ss}\boldsymbol{\Phi}^T + \mathbf{Q}, \tag{32}$$

where $\bar{\mathbf{R}} = \mathbf{H}\mathbf{P}_{ss}\mathbf{H}^T + r\mathbf{I}_3$. Consequently, the gain $\mathbf{K}_k$ in (27) converges to a constant steady-state value as well, given by

$$\mathbf{K}_{ss} = \boldsymbol{\Phi}\mathbf{P}_{ss}\mathbf{H}^T\bar{\mathbf{R}}^{-1}. \tag{33}$$

It can be readily shown that the matrix $\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H}$ is stable. As such, using the control-like input $\breve{\mathbf{u}}_k = \mathbf{K}_{ss}\mathbf{y}_k$ would yield a stable filter. In addition, it is well-known that the steady-state behavior of such a filter is the same as that for the time-varying optimal Kalman filter (Chui and Chen (1999)). The advantage of using the time-varying optimal Kalman filter gain, over the constant steady-state optimal Kalman gain, is better transient performance. To partially recover the rapid transient performance of the optimal Kalman filter, a simple analytical approximation to the Kalman gain is sought, of the form

$$\bar{\mathbf{K}}_k = \begin{cases} \tilde{\mathbf{K}}_k, & 0 \leq k \leq k^* \\ \mathbf{K}_{ss}, & k > k^* \end{cases} \tag{34}$$

such that $\tilde{\mathbf{K}}_k \approx \mathbf{K}_k$ for $0 \leq k \leq k^*$, where $\mathbf{K}_k$ is the optimal Kalman gain, satisfying (27) (with the matrices in (29) and (31)), and $\mathbf{K}_{ss}$ is a constant gain. The quantity k^* is the instant at which the gain is switched from time-varying to constant, and reflects the transition from transient to steady-state filter operation. Note that $\mathbf{K}_{ss}$ in (34) does not need to satisfy (33), but can be chosen according to other criteria, such as robustness. The control-like input corresponding to (34) is

$$\breve{\mathbf{u}}_k = \bar{\mathbf{K}}_k\mathbf{y}_k. \tag{35}$$

To further facilitate the development of analytical approximations to the optimal Kalman gain, it is additionally assumed that the initial estimation error covariance has the form

$$\mathbf{P}_0 = \begin{bmatrix} \sigma_r\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \sigma_v\mathbf{I}_3 \end{bmatrix}, \tag{36}$$

where $\sigma_r > 0$ and $\sigma_v > 0$.

With the matrices given in (29), (31) and (36), the

Kalman gain matrix in (27) takes the form

$$\mathbf{K}_k = \begin{bmatrix} k_{r,k}\mathbf{I}_3 & k_{v,k}\mathbf{I}_3 \end{bmatrix}^T \tag{37}$$

where

$$\hat{\mathbf{K}}_k = \begin{bmatrix} k_{r,k} & k_{v,k} \end{bmatrix}^T = \hat{\mathbf{\Phi}}\hat{\mathbf{P}}_k\hat{\mathbf{H}}^T(\hat{\mathbf{H}}\hat{\mathbf{P}}_k\hat{\mathbf{H}}^T + r)^{-1} \tag{38}$$

is the Kalman gain corresponding to the decoupled system in (21), with the corresponding error covariance for the decoupled system satisfying

$$\hat{\mathbf{P}}_{k+1} = \hat{\mathbf{\Phi}}\hat{\mathbf{P}}_k\hat{\mathbf{\Phi}}^T - \hat{\mathbf{\Phi}}\hat{\mathbf{P}}_k\hat{\mathbf{H}}^T(\hat{\mathbf{H}}\hat{\mathbf{P}}_k\hat{\mathbf{H}}^T + r)^{-1}\hat{\mathbf{H}}\hat{\mathbf{P}}_k\hat{\mathbf{\Phi}}^T + \hat{\mathbf{Q}}, \tag{39}$$

where

$$\hat{\mathbf{P}}_0 = \begin{bmatrix} \sigma_r & 0 \\ 0 & \sigma_v \end{bmatrix}, \quad \hat{\mathbf{Q}} = \begin{bmatrix} q_{rr} & q_{rv} \\ q_{rv} & q_{vv} \end{bmatrix}, \tag{40}$$

and all other system matrices are given in (22). Likewise, the estimation error covariance for the overall system takes the form

$$\mathbf{P}_k = \begin{bmatrix} p_{rr,k}\mathbf{I}_3 & p_{rv,k}\mathbf{I}_3 \\ p_{rv,k}\mathbf{I}_3 & p_{vv,k}\mathbf{I}_3 \end{bmatrix}, \tag{41}$$

where

$$\hat{\mathbf{P}}_k = \begin{bmatrix} p_{rr,k} & p_{rv,k} \\ p_{rv,k} & p_{vv,k} \end{bmatrix}, \tag{42}$$

satisfies (39).

3.1 Analytical Approximation of the Optimal Transient Gain

Under the assumption that the initial estimation error covariance in (36) satisfies $\mathbf{P}_0 >> \mathbf{P}_{ss}$, where $\mathbf{P}_{ss}$ is the positive definite solution of (32), the analytical approximations to the optimal Kalman filter gains $k_{r,k}$ and $k_{v,k}$ in (38) are (see Appendix A for derivation)

$$\begin{aligned} \tilde{k}_{r,k} &= \frac{(T^2/(6r))(k-k^3) + 1/\sigma_v}{rd_k} + (k+1)T\tilde{k}_{v,k}, \\ \tilde{k}_{v,k} &= \frac{(1/r)(1-T/2)k^2 + ((1/r)(1-T/2) + 1/\sigma_r)\,k}{rd_k}, \end{aligned} \tag{43}$$

respectively, where

$$\begin{aligned} d_k = {} & \frac{T^2}{12r^2}k^4 + \frac{T^2}{3r}\left(\frac{1}{r} + \frac{1}{\sigma_r}\right)k^3 + \frac{T^2}{2r}\left(\frac{5}{6r} + \frac{1}{\sigma_r}\right)k^2 \\ & + \left(\frac{T^2}{6r}\left(\frac{1}{r} + \frac{1}{\sigma_r}\right) + \frac{1}{r\sigma_v}\right)k + \frac{1}{\sigma_v}\left(\frac{1}{r} + \frac{1}{\sigma_r}\right). \end{aligned}$$

The switching time k^* in (34) is given by

$$k^* = \max\{k_1^*, k_2^*\}, \tag{44}$$

where

$$k_1^* = \min\{k : \frac{k+1}{r} \geq \frac{\chi}{\sigma_r}\}, \quad k_2^* = \min\{k : \frac{k^3T^2}{3r} \geq \frac{\chi}{\sigma_v}\}, \tag{45}$$

where $\chi >> 1$ is a user-specified parameter.

3.2 Steady-State Gain, Stability and Performance

The design of the steady-state part of the gain in (34) is now considered, stability is analyzed and bounds on the steady-state performance are obtained, when applied to the actual system in (16). Since the time-varying transient gain in (34) is applied only for a pre-determined fixed period of time, the analyses of the stability under the steady-state gain as well as the limiting steady-state performance can occur independently from the time-period of application of the transient gain. Therefore in the following, the time index k is reset to $k = 0$ at the instant $k = k^*$.

Consider any constant gain

$$\hat{\mathbf{K}}_{ss} = \begin{bmatrix} k_{r,ss} & k_{v,ss} \end{bmatrix}^T, \tag{46}$$

such that $\hat{\mathbf{\Phi}} - \hat{\mathbf{K}}_{ss}\hat{\mathbf{H}}$ is a stable matrix. Then, setting

$$\mathbf{K}_{ss} = \begin{bmatrix} k_{r,ss}\mathbf{I}_3 & k_{v,ss}\mathbf{I}_3 \end{bmatrix}^T, \tag{47}$$

the matrix $\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H}$ is also stable. Application of the control-like input $\breve{\mathbf{u}}_k = \mathbf{K}_{ss}\mathbf{y}_k$ to the ideal system in (14), leads to the ideal closed-loop system

$$\mathbf{x}_{k+1} = (\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H})\mathbf{x}_k - \mathbf{K}_{ss}\mathbf{n}_k + \mathbf{w}_k, \tag{48}$$

with corresponding decoupled closed-loop system

$$\mathbf{z}_{k+1,i} = (\hat{\mathbf{\Phi}} - \hat{\mathbf{K}}_{ss}\hat{\mathbf{H}})\mathbf{z}_{k,i} - \hat{\mathbf{K}}_{ss}n_{k,i} + \mathbf{w}_{k,i}, \quad i = 1,2,3. \tag{49}$$

On the other hand, application of the control-like input $\breve{\mathbf{u}}_k = \mathbf{K}_{ss}\mathbf{y}_k$ to the real system in (16) leads to the real closed-loop system

$$\mathbf{x}_{k+1} = (\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H} + \bar{\mathbf{\Phi}}_k)\mathbf{x}_k - (\mathbf{I}_6 + \tilde{\mathbf{B}}_k)\mathbf{K}_{ss}\mathbf{n}_k + \mathbf{w}_k, \tag{50}$$

where

$$\bar{\mathbf{\Phi}}_k = \tilde{\mathbf{\Phi}}_k - \tilde{\mathbf{B}}_k\mathbf{K}_{ss}\mathbf{H}. \tag{51}$$

There are now three closed-loop systems of interest. The first is the ideal closed-loop system (48) with measurement and process noise covariances given by (31). The corresponding estimation error covariance will be denoted $\mathbf{P}_{I,k}$. The second is the actual closed-loop system (50) with measurement and process noise covariances also given by (31). The corresponding estimation error covariance will be denoted $\mathbf{P}_{A,k}$. The third is the actual closed-loop system with the true measurement and process noise covariances corresponding to (1) and (7). The corresponding estimation error covariance will be denoted $\mathbf{P}_{T,k}$. These covariances satisfy the recursions (Chui and Chen (1999))

$$\begin{aligned}\mathbf{P}_{I,k+1} &= (\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H})\mathbf{P}_{I,k}(\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H})^T \\ &\quad + r\mathbf{K}_{ss}\mathbf{K}_{ss}^T + \mathbf{Q}, \qquad (52)\\ \mathbf{P}_{A,k+1} &= (\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H} + \bar{\boldsymbol{\Phi}}_k)\mathbf{P}_{A,k}(\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H} + \bar{\boldsymbol{\Phi}}_k)^T \\ &\quad + r(\mathbf{I}_6 + \tilde{\mathbf{B}}_k)\mathbf{K}_{ss}\mathbf{K}_{ss}^T(\mathbf{I}_6 + \tilde{\mathbf{B}}_k)^T + \mathbf{Q}, \qquad (53)\\ \mathbf{P}_{T,k+1} &= (\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H} + \bar{\boldsymbol{\Phi}}_k)\mathbf{P}_{T,k}(\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H} + \bar{\boldsymbol{\Phi}}_k)^T \\ &\quad + (\mathbf{I}_6 + \tilde{\mathbf{B}}_k)\mathbf{K}_{ss}\mathbf{R}_k\mathbf{K}_{ss}^T(\mathbf{I}_6 + \tilde{\mathbf{B}}_k)^T + \mathbf{Q}_k. \qquad (54)\end{aligned}$$

Since $\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H}$ is a stable matrix, $\mathbf{P}_{I,k}$ in (52) approaches a constant steady-state positive-definite matrix $\mathbf{P}_{I,ss}$, satisfying (Chui and Chen (1999))

$$\mathbf{P}_{I,ss} = (\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H})\mathbf{P}_{I,ss}(\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H})^T + r\mathbf{K}_{ss}\mathbf{K}_{ss}^T + \mathbf{Q}. \qquad (55)$$

The limit in (55) is approached by $\mathbf{P}_{I,k}$ regardless of the initial condition $\mathbf{P}_{I,0}$. Furthermore, given the structure of $\boldsymbol{\Phi}$, $\mathbf{K}_{ss}$, $\mathbf{H}$ and $\mathbf{Q}$, it is readily verified that

$$\mathbf{P}_{I,ss} = \begin{bmatrix} p_{rr}\mathbf{I}_3 & p_{rv}\mathbf{I}_3 \\ p_{rv}\mathbf{I}_3 & p_{vv}\mathbf{I}_3 \end{bmatrix}, \quad \hat{\mathbf{P}}_{I,ss} = \begin{bmatrix} p_{rr} & p_{rv} \\ p_{rv} & p_{vv} \end{bmatrix}, \qquad (56)$$

where

$$\hat{\mathbf{P}}_{I,ss} = (\hat{\boldsymbol{\Phi}} - \hat{\mathbf{K}}_{ss}\hat{\mathbf{H}})\hat{\mathbf{P}}_{I,ss}(\hat{\boldsymbol{\Phi}} - \hat{\mathbf{K}}_{ss}\hat{\mathbf{H}})^T + r\hat{\mathbf{K}}_{ss}\hat{\mathbf{K}}_{ss}^T + \hat{\mathbf{Q}}. \qquad (57)$$

From equations (53) and (54), it is readily seen that if (30) holds, and if $\mathbf{P}_{A,0} \geq \mathbf{P}_{T,0}$, then

$$\mathbf{P}_{A,k} \geq \mathbf{P}_{T,k}, \ \ \forall k \geq 0. \qquad (58)$$

Consequently, any bounds obtained on the limiting behavior of $\mathbf{P}_{A,k}$ are automatically bounds on the limiting behavior of $\mathbf{P}_{T,k}$. The steady-state performance analysis therefore focuses on $\mathbf{P}_{A,k}$, whose governing difference equations in (53) are a perturbation of the difference equations governing $\mathbf{P}_{I,k}$ in (52), for which the limiting behavior has been established in equations (55) to (57).

It will be useful to consider the change of coordinates

$$\bar{\mathbf{x}}_k = \mathbf{S}\mathbf{x}_k, \ \ \mathbf{S} = \mathrm{diag}\{\mathbf{I}_3, \alpha_v\mathbf{I}_3\}, \qquad (59)$$

and $\alpha_v > 0$ is a velocity scaling factor. Associated with (59), define the matrix

$$\hat{\mathbf{S}} = \mathrm{diag}\{1, \alpha_v\}. \qquad (60)$$

Theorem 1 *Consider $\mathbf{K}_{ss}$ as given in equation (47), such that $\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H}$ has eigenvalues inside the open unit disk. Furthermore, suppose that $\|\boldsymbol{\omega}_x\| \leq \bar{\omega}$ and $\|\partial\mathbf{a}_g(\hat{\mathbf{r}}(t), t)/\partial\mathbf{r}\| \leq \bar{g}$, where $\bar{\omega}$ and $\bar{g}$ are known positive constants. Then, $\mathbf{x}_{k+1} = (\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H} + \bar{\boldsymbol{\Phi}}_k)\mathbf{x}_k$ is exponentially stable if there exists a symmetric positive-definite matrix $\mathbf{Z} \in \mathbb{R}^{2\times 2}$ such that*

$$\gamma_1 < \lambda_{\min}(\mathbf{Z}), \qquad (61)$$

where

$$\begin{aligned}\gamma_1 &= \bar{g}\|\hat{\mathbf{S}}^{-1}\mathbf{A}_1^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}\hat{\boldsymbol{\Phi}}_{cl}\hat{\mathbf{S}}^{-1} + \hat{\mathbf{S}}^{-1}\hat{\boldsymbol{\Phi}}_{cl}^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}\mathbf{A}_1\hat{\mathbf{S}}^{-1}\| \\ &\quad + \bar{\omega}|X_{12} - X_{21}| + \bar{g}^2\|\hat{\mathbf{S}}^{-1}\mathbf{A}_1^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}\mathbf{A}_1\hat{\mathbf{S}}^{-1}\| \\ &\quad + \bar{\omega}^2\|\hat{\mathbf{S}}^{-1}\mathbf{A}_2^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}\mathbf{A}_2\hat{\mathbf{S}}^{-1}\| \\ &\quad + 2\bar{g}\bar{\omega}\|\hat{\mathbf{S}}^{-1}\mathbf{A}_2^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}\mathbf{A}_1\hat{\mathbf{S}}^{-1}\|, \qquad (62)\end{aligned}$$

where $\hat{\boldsymbol{\Phi}}_{cl} = \hat{\boldsymbol{\Phi}} - \hat{\mathbf{K}}_{ss}\hat{\mathbf{H}}$, and $\mathbf{W} \in \mathbb{R}^{2\times 2}$ is the unique symmetric positive-definite solution of

$$\hat{\mathbf{S}}^{-1}\hat{\boldsymbol{\Phi}}_{cl}^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}\hat{\boldsymbol{\Phi}}_{cl}\hat{\mathbf{S}}^{-1} - \mathbf{W} = -\mathbf{Z}, \qquad (63)$$

and the remaining matrices are given by

$$\mathbf{A}_1 = \begin{bmatrix} T^2\,(1/2 - k_{r,ss}/6 + k_{v,ss}T/24) & T^3/6 \\ T\,(1 - k_{r,ss}/2 + k_{v,ss}T/12) & T^2/2 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} (1 - k_{r,ss}/2)T - k_{v,ss}T^2/12 & T^2 \\ -k_{v,ss}T/2 & T \end{bmatrix}, \qquad (64)$$

and

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \hat{\mathbf{S}}^{-1}\hat{\boldsymbol{\Phi}}_{cl}^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}\mathbf{A}_2\hat{\mathbf{S}}^{-1}. \qquad (65)$$

Proof See Appendix B.

Theorem 2 *Consider the difference*

$$\tilde{\mathbf{P}}_k = \mathbf{P}_{A,k} - \mathbf{P}_{I,ss}, \qquad (66)$$

where $\mathbf{P}_{A,k}$ satisfies the recursion in (53), and $\mathbf{P}_{I,ss}$ is the unique positive-definite solution of (55). Suppose that all of the conditions in Theorem 1 hold, and that (61) is satisfied. Then,

$$\limsup_{k\to\infty} \|\mathbf{S}\tilde{\mathbf{P}}_k\mathbf{S}\| \leq \frac{\lambda_{\max}(\mathbf{W})^2}{\lambda_{\min}(\mathbf{W})}\frac{\gamma_2}{\lambda_{\min}(\mathbf{W}) - \gamma_1}, \qquad (67)$$

where

$$\gamma_2 = \bar{g}\|\hat{\mathbf{S}}\mathbf{A}_1\hat{\mathbf{P}}_{I,ss}\hat{\boldsymbol{\Phi}}_{cl}^T\hat{\mathbf{S}} + \hat{\mathbf{S}}\hat{\boldsymbol{\Phi}}_{cl}\hat{\mathbf{P}}_{I,ss}\mathbf{A}_1^T\hat{\mathbf{S}}\| \\ +\bar{g}^2\|\hat{\mathbf{S}}\mathbf{A}_1\hat{\mathbf{P}}_{I,ss}\mathbf{A}_1^T\hat{\mathbf{S}}\| + \bar{\omega}^2\|\hat{\mathbf{S}}\mathbf{A}_2\hat{\mathbf{P}}_{I,ss}\mathbf{A}_2^T\hat{\mathbf{S}}\| \\ +2\bar{g}\bar{\omega}\|\hat{\mathbf{S}}\mathbf{A}_2\hat{\mathbf{P}}_{I,ss}\mathbf{A}_1^T\hat{\mathbf{S}}\| + 2r\bar{g}\bar{\omega}\|\hat{\mathbf{S}}\mathbf{F}_2\hat{\mathbf{K}}_{ss}\hat{\mathbf{K}}_{ss}^T\mathbf{F}_1^T\hat{\mathbf{S}}\| \\ +r\bar{g}\|\hat{\mathbf{S}}\mathbf{F}_1\hat{\mathbf{K}}_{ss}\hat{\mathbf{K}}_{ss}^T\hat{\mathbf{S}} + \hat{\mathbf{S}}\hat{\mathbf{K}}_{ss}\hat{\mathbf{K}}_{ss}^T\mathbf{F}_1^T\hat{\mathbf{S}}\| \\ +r\bar{\omega}^2\|\hat{\mathbf{S}}\mathbf{F}_2\hat{\mathbf{K}}_{ss}\hat{\mathbf{K}}_{ss}^T\mathbf{F}_2^T\hat{\mathbf{S}}\| + r\bar{g}^2\|\hat{\mathbf{S}}\mathbf{F}_1\hat{\mathbf{K}}_{ss}\hat{\mathbf{K}}_{ss}^T\mathbf{F}_1^T\hat{\mathbf{S}}\| \\ +\bar{\omega}|Y_{12} - Y_{21}| + r\bar{\omega}|L_{12} - L_{21}|, \tag{68}$$

with

$$\mathbf{F}_1 = \begin{bmatrix} T^2/6 & -T^3/24 \\ T/2 & -T^2/12 \end{bmatrix}, \quad \mathbf{F}_2 = \begin{bmatrix} T/2 & T^2/12 \\ 0 & T/2 \end{bmatrix}, \tag{69}$$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \hat{\mathbf{S}}\mathbf{A}_2\hat{\mathbf{P}}_{I,ss}\hat{\boldsymbol{\Phi}}_{cl}^T\hat{\mathbf{S}}, \tag{70}$$

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \hat{\mathbf{S}}\mathbf{F}_2\hat{\mathbf{K}}_{ss}\hat{\mathbf{K}}_{ss}^T\hat{\mathbf{S}}, \tag{71}$$

and all remaining parameters come from the statement of Theorem 1.

Proof See Appendix C.

The result from Theorem 2 can be used to obtain bounds on the steady-state performance of the position and velocity estimation errors separately. First of all, partitioning

$$\tilde{\mathbf{P}}_k = \begin{bmatrix} \tilde{\mathbf{P}}_{rr,k} & \tilde{\mathbf{P}}_{rv,k} \\ \tilde{\mathbf{P}}_{rv,k}^T & \tilde{\mathbf{P}}_{vv,k} \end{bmatrix}, \tag{72}$$

where $\tilde{\mathbf{P}}_{rr,k}, \tilde{\mathbf{P}}_{rv,k}, \tilde{\mathbf{P}}_{vv,k} \in \mathbb{R}^{3\times 3}$, equation (59) yields

$$\mathbf{S}\tilde{\mathbf{P}}_k\mathbf{S} = \begin{bmatrix} \tilde{\mathbf{P}}_{rr,k} & \alpha_v\tilde{\mathbf{P}}_{rv,k} \\ \alpha_v\tilde{\mathbf{P}}_{rv,k}^T & \alpha_v^2\tilde{\mathbf{P}}_{vv,k} \end{bmatrix}. \tag{73}$$

Consequently, for any of the position estimation errors, one obtains for $i = 1, 2, 3$,

$$\limsup_{k\to\infty} E\{\tilde{r}_i(t_k)^2\} \leq p_{rr} + \frac{\lambda_{\max}(\mathbf{W})^2}{\lambda_{\min}(\mathbf{W})}\frac{\gamma_2}{\lambda_{\min}(\mathbf{W}) - \gamma_1}, \\ \limsup_{k\to\infty} E\{\tilde{v}_i(t_k)^2\} \leq p_{vv} + \frac{1}{\alpha_v^2}\frac{\lambda_{\max}(\mathbf{W})^2}{\lambda_{\min}(\mathbf{W})}\frac{\gamma_2}{\lambda_{\min}(\mathbf{W}) - \gamma_1}, \tag{74}$$

where (58), (66), (56), (73) and (67) have been used (in that order), together with the fact that any element $\{\mathbf{X}\}_{i,j}$ of the matrix $\mathbf{X}$, satisfies $|\{\mathbf{X}\}_{i,j}| \leq \|\mathbf{X}\|$. Note that (67) holds independently of the initial condition $\mathbf{P}_{A,0}$. Therefore, one can always choose an initial $\mathbf{P}_{A,0}$ such that (58) holds. The usefulness of the velocity scaling factor α_v is now apparent from (74). It allows one to separate the bounds on the steady-state position and velocity estimation performances. In particular, $\mathbf{Z} > \mathbf{0}$, $\alpha_v > 0$ are free parameters that may be optimized (subject to (61)), to separately obtain the tightest bounds on the steady-state position and velocity estimation performances, respectively in (74).

3.3 Filter Summary

Form the filter correction

$$\begin{bmatrix} \bar{\mathbf{u}}_{r,k} \\ \bar{\mathbf{u}}_{v,k} \end{bmatrix} = \begin{bmatrix} (1/T)\mathbf{I}_3 & -(1/2)\mathbf{I}_3 \\ \mathbf{0}_3 & (1/T)\mathbf{I}_3 \end{bmatrix} \begin{bmatrix} k_{r,k}\mathbf{I}_3 \\ k_{v,k}\mathbf{I}_3 \end{bmatrix} \mathbf{y}_k,$$

where $\mathbf{y}_k = \mathbf{r}^m(t_k) - \hat{\mathbf{r}}(t_k)$ and $(k_{r,k}, k_{v,k}) = (\tilde{k}_{r,k}, \tilde{k}_{v,k})$ from (43) if $k \leq k^*$, and $(k_{r,k}, k_{v,k}) = (k_{r,ss}, k_{v,ss})$ if $k > k^*$. Setting $\mathbf{u}(t) = \bar{\mathbf{u}}_k$, integrate (3) from t_k to t_{k+1}.

Remark *While the presented filter is for absolute motion estimation, the relative motion equations for a pair of spacecraft flying in close proximity take the same form, with one set of motion equations for the absolute motion of one of the spacecraft, and the second set of motion equations for the relative motion of the second spacecraft relative to the first. Further details can be found in de Ruiter et al. (2008). As such, the developed filter can be applied to spacecraft relative motion estimation also.*

4 Numerical Example

In this example, a pair of spacecraft are flying in a circular Earth orbit, at an altitude of 650 km in a close along-track formation with initial separation of 15 km. The absolute and relative position measurements are provided by a pair of GPS receivers, located one on each spacecraft. The reference frame $\mathcal{F}_x$ within which filtering is performed is the Earth-Centered-Earth-Fixed frame. Details of the pre-processing of the GPS measurements may be found in de Ruiter et al. (2008). The proposed filter is implemented in a very high fidelity hardware-in-the-loop simulation, including two real GPS receivers and a GPS simulator. Further details of the hardware-in-the-loop setup and the simulation parameters may be found in de Ruiter et al. (2008).

The proposed suboptimal filter is compared to a full extended Kalman filter (EKF), which is applied to the full set of spacecraft formation equations (both absolute and relative motion), without separating absolute from relative motion estimation. To make the comparison fair, the EKF uses the same measurements as the sub-optimal filter. Figure 1 shows the Euclidean norm of the relative position estimation errors. It can be seen that the proposed sub-optimal filter achieves very similar performance to the EKF, both in transient and steady-state. Similar results (not shown) hold for the relative velocity error.

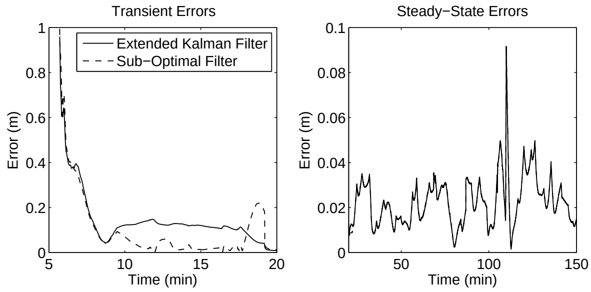

Fig. 1. Relative Position Errors

5 Concluding Remarks

A computationally simple near-optimal filter has been presented for spacecraft motion estimation utilizing position measurements. Unlike the extended Kalman filter, it requires two scalar gains and no covariance propagation. Each gain is separated into two parts: a time-varying transient part, and a constant steady-state part. The time-varying part is a simple analytical approximation to the optimal Kalman gain for a double integrator system with position measurement, and is applied for a predetermined fixed period of time. After this time, the gain is switched to the constant steady-state value. Analytical steady-state performance bounds are derived for the filter, allowing the user to predict the filter steady-state performance when applied to the actual space system. The efficacy of the proposed filter is demonstrated by a hardware-in-the-loop simulation for relative navigation of a two-spacecraft formation. It is seen that the proposed filter obtains very similar performance to a full extended Kalman filter for the same problem, both in transient and in steady-state.

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A Derivation of Transient Gains

The derivation of the discrete-time transient gains here is similar in spirit to the continuous-time case in Kim (1990), de Ruiter (2012). Analytical approximations are sought for the gains $k_{r,k}$ and $k_{v,k}$, which satisfy equations (38) and (39). Denoting the steady-state solution of (39) by $\hat{\mathbf{P}}_{ss}$, it is assumed that the initial condition $\hat{\mathbf{P}}_0$ of (39) is relatively large, such that $\hat{\mathbf{P}}_0 >> \hat{\mathbf{P}}_{ss}$. Under this assumption, it is reasonable to assume that the solution of (39) is initially driven by the initial condition $\hat{\mathbf{P}}_0$, rather than the process noise with covariance $\hat{\mathbf{Q}}$. As such, the transient gain is obtained by solving (39) with $\hat{\mathbf{Q}} = \mathbf{0}$, and the gain in (38) is found to be

$$\hat{\mathbf{K}}_k = \frac{1}{r}\hat{\mathbf{\Phi}}^{k+1}\left(\hat{\mathbf{P}}_0^{-1} + \frac{1}{r}\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i\right)^{-1} \times \left(\hat{\mathbf{\Phi}}^T\right)^k \hat{\mathbf{H}}^T, \tag{A.1}$$

for $k \geq 0$. From (22), and the fact that (Edwards (1986))

$$\sum_{i=0}^{k} i = k(k+1)/2, \quad \sum_{i=0}^{k} i^2 = k(k+1)(2k+1)/6,$$

the summation in (A.1) becomes

$$\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i = \begin{bmatrix} k+1 & \frac{k(k+1)T}{2} \\ \frac{k(k+1)T}{2} & \frac{k(k+1)(2k+1)T^2}{6} \end{bmatrix}. \tag{A.2}$$

Substituting $\hat{\mathbf{P}}_0$ together with (22) and (A.2) into (A.1) and expanding yields the approximations given in (43). From the gain expression in (A.1), $\hat{\mathbf{P}}_0$ enters through the term $\hat{\mathbf{P}}_0^{-1} + (1/r)\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i$. It can be seen that this term is initially dominated by the initial condition $\hat{\mathbf{P}}_0^{-1}$, and later dominated by the summation $(1/r)\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i$. Consider the times

$$k_j^* = \min\left\{k : \frac{1}{r}\left\{\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i\right\}_{jj} \geq \chi/\sigma_j\right\}, \tag{A.3}$$

for $j = 1, 2$ where $\chi >> 1$. The time k_j^* then reflects the time when σ_j ceases to be influential in the jj^{th} term of $\hat{\mathbf{P}}_0^{-1} + (1/r)\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i$. To simplify matters further, from (A.2), one has approximately

$$\frac{1}{r}\left\{\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i\right\}_{jj} \approx \begin{cases} (k+1)/r, & j=1, \\ k^3T^2/(3r), & j=2. \end{cases} \tag{A.4}$$

A suitable time to switch from transient to steady-state operation is when both diagonal terms in $\hat{\mathbf{P}}_0^{-1} + (1/r)\sum_{i=0}^{k}\left(\hat{\mathbf{\Phi}}^T\right)^i \hat{\mathbf{H}}^T\hat{\mathbf{H}}\hat{\mathbf{\Phi}}^i$ are dominated by the summation term. Using equations (A.3) and (A.4), the expressions given in (44) and (45) are obtained.

B Proof of Theorem 1

Application of the coordinate transformation in (59) yields

$$\bar{\mathbf{x}}_{k+1} = \left(\mathbf{S}(\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H})\mathbf{S}^{-1} + \mathbf{S}\bar{\mathbf{\Phi}}_k\mathbf{S}^{-1}\right)\bar{\mathbf{x}}_k. \tag{B.1}$$

Since $\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H}$ has eigenvalues inside the open unit disk, given any symmetric positive definite matrix $\bar{\mathbf{Z}} \in \mathbb{R}^{6\times6}$, the discrete-time Lyapunov equation

$$\mathbf{S}^{-1}(\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H})^T\mathbf{S}\bar{\mathbf{W}}\mathbf{S}(\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H})\mathbf{S}^{-1} - \bar{\mathbf{W}} = -\bar{\mathbf{Z}}, \tag{B.2}$$

has a unique symmetric positive-definite solution $\bar{\mathbf{W}}$ (Zhou et al. (1996)). In particular, choose $\bar{\mathbf{Z}} \in \mathbb{R}^6$ to have the form

$$\bar{\mathbf{Z}} = \mathbf{T}^T\mathrm{diag}\{\mathbf{Z}, \mathbf{Z}, \mathbf{Z}\}\mathbf{T}, \tag{B.3}$$

where $\mathbf{T}$ is given in equation (20) and $\mathbf{Z} \in \mathbb{R}^{2\times2}$ is symmetric and positive-definite. As a consequence, the unique solution $\bar{\mathbf{W}}$ of (B.2) is given by

$$\bar{\mathbf{W}} = \mathbf{T}^T\begin{bmatrix} \mathbf{W} & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{W} & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{W} \end{bmatrix}\mathbf{T} = \begin{bmatrix} w_{11}\mathbf{I}_3 & w_{12}\mathbf{I}_3 \\ w_{12}\mathbf{I}_3 & w_{22}\mathbf{I}_3 \end{bmatrix}, \tag{B.4}$$

where

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}, \tag{B.5}$$

is the unique symmetric positive-definite solution of

$$\hat{\mathbf{S}}^{-1}(\hat{\mathbf{\Phi}} - \hat{\mathbf{K}}_{ss}\hat{\mathbf{H}})^T\hat{\mathbf{S}}\mathbf{W}\hat{\mathbf{S}}(\hat{\mathbf{\Phi}} - \hat{\mathbf{K}}_{ss}\hat{\mathbf{H}})\hat{\mathbf{S}}^{-1} - \mathbf{W} = -\mathbf{Z}. \tag{B.6}$$

Now, consider the Lyapunov-function candidate

$$V_k = \bar{x}_k^T\bar{\mathbf{W}}\bar{\mathbf{x}}_k. \tag{B.7}$$

Utilizing (B.1) and (B.2), one obtains

$$V_{k+1} - V_k = -\bar{\mathbf{x}}_k^T\bar{\mathbf{Z}}\bar{\mathbf{x}}_k + \bar{\mathbf{x}}_k^T\tilde{\mathbf{Z}}\bar{\mathbf{x}}_k, \tag{B.8}$$

where

$$\begin{aligned}\tilde{\mathbf{Z}} = {} & \mathbf{S}^{-1}\bar{\mathbf{\Phi}}_k^T\mathbf{S}\bar{\mathbf{W}}\mathbf{S}(\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H})\mathbf{S}^{-1} \\ & +\mathbf{S}^{-1}(\mathbf{\Phi} - \mathbf{K}_{ss}\mathbf{H})^T\mathbf{S}\bar{\mathbf{W}}\mathbf{S}\bar{\mathbf{\Phi}}_k\mathbf{S}^{-1} \\ & +\mathbf{S}^{-1}\bar{\mathbf{\Phi}}_k^T\mathbf{S}\bar{\mathbf{W}}\mathbf{S}\bar{\mathbf{\Phi}}_k\mathbf{S}^{-1},\end{aligned} \tag{B.9}$$

which after some work can be bounded by

$$\|\tilde{\mathbf{Z}}\| \le \gamma_1, \tag{B.10}$$

where γ_1 is given in (62). Next, from (B.7), (B.4) and (B.5) and the fact that $\mathbf{W}$ is symmetric and positive-definite,

$$\lambda_{\min}(\mathbf{W})\|\bar{\mathbf{x}}_k\|^2 \le V_k(\bar{\mathbf{x}}_k) \le \lambda_{\max}(\mathbf{W})\|\bar{\mathbf{x}}_k\|^2. \tag{B.11}$$

Consequently, after some work (B.8) leads to

$$V_{k+1} \le \left(1 - \frac{\lambda_{\min}(\mathbf{Z}) - \gamma_1}{\lambda_{\max}(\mathbf{W})}\right) V_k. \tag{B.12}$$

The conclusion of Theorem 1 follows directly from (B.12).

C Proof of Theorem 2

Substitution of (66) into (53), and using (55) leads to the recursion for $\tilde{\mathbf{P}}_k$

$$\tilde{\mathbf{P}}_k = \breve{\boldsymbol{\Phi}}_k \tilde{\mathbf{P}}_k \breve{\boldsymbol{\Phi}}_k^T + \bar{\mathbf{Q}}_k, \tag{C.1}$$

where

$$\breve{\boldsymbol{\Phi}}_k = \boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H} + \bar{\boldsymbol{\Phi}}_k, \tag{C.2}$$

and

$$\begin{aligned}\bar{\mathbf{Q}}_k = {} & \bar{\boldsymbol{\Phi}}_k \mathbf{P}_{I,ss} \bar{\boldsymbol{\Phi}}_k^T + \bar{\boldsymbol{\Phi}}_k \mathbf{P}_{I,ss}(\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H})^T \\ & + (\boldsymbol{\Phi} - \mathbf{K}_{ss}\mathbf{H})\mathbf{P}_{I,ss}\bar{\boldsymbol{\Phi}}_k^T + r\tilde{\mathbf{B}}_k \mathbf{K}_{ss}\mathbf{K}_{ss}^T \\ & + r\mathbf{K}_{ss}\mathbf{K}_{ss}^T \tilde{\mathbf{B}}_k^T + r\tilde{\mathbf{B}}_k \mathbf{K}_{ss}\mathbf{K}_{ss}^T \tilde{\mathbf{B}}_k^T. \end{aligned} \tag{C.3}$$

The recursion in (C.1) has solution

$$\tilde{\mathbf{P}}_k = \breve{\boldsymbol{\Phi}}_{k,0}\tilde{\mathbf{P}}_0\breve{\boldsymbol{\Phi}}_{k,0}^T + \sum_{i=0}^{k-1} \breve{\boldsymbol{\Phi}}_{k-1,i+1}\bar{\mathbf{Q}}_i\breve{\boldsymbol{\Phi}}_{k-1,i+1}^T, \quad k \ge 1, \tag{C.4}$$

where

$$\breve{\boldsymbol{\Phi}}_{j,i} = \breve{\boldsymbol{\Phi}}_j \breve{\boldsymbol{\Phi}}_{j-1} \cdots \breve{\boldsymbol{\Phi}}_{i+1}\breve{\boldsymbol{\Phi}}_i, \quad \breve{\boldsymbol{\Phi}}_{i,i+1} = \mathbf{I}_6, \quad j \ge i \ge 0, \tag{C.5}$$

have been defined. Equation (C.4) leads to

$$\begin{aligned}\mathbf{S}\tilde{\mathbf{P}}_k\mathbf{S} = {} & (\mathbf{S}\breve{\boldsymbol{\Phi}}_{k,0}\mathbf{S}^{-1})\mathbf{S}\tilde{\mathbf{P}}_0\mathbf{S}(\mathbf{S}^{-1}\breve{\boldsymbol{\Phi}}_{k,0}^T\mathbf{S}) \\ & + \sum_{i=0}^{k-1}(\mathbf{S}\breve{\boldsymbol{\Phi}}_{k-1,i+1}\mathbf{S}^{-1})(\mathbf{S}\bar{\mathbf{Q}}_i\mathbf{S})(\mathbf{S}^{-1}\breve{\boldsymbol{\Phi}}_{k-1,i+1}^T\mathbf{S}),\end{aligned} \tag{C.6}$$

for $k \ge 1$. Taking the norms of both sides of (C.6) leads to

$$\begin{aligned}\|\mathbf{S}\tilde{\mathbf{P}}_k\mathbf{S}\| \le {} & \|\mathbf{S}\breve{\boldsymbol{\Phi}}_{k,0}\mathbf{S}^{-1}\|^2\|\mathbf{S}\tilde{\mathbf{P}}_0\mathbf{S}\| \\ & + \sum_{i=0}^{k-1}\|\mathbf{S}\breve{\boldsymbol{\Phi}}_{k-1,i+1}\mathbf{S}^{-1}\|^2\|\mathbf{S}\bar{\mathbf{Q}}_i\mathbf{S}\|,\end{aligned} \tag{C.7}$$

for $k \ge 1$. Using the symmetry of $\partial\mathbf{a}_g(\mathbf{r},t)/\partial\mathbf{r}$, and the skew-symmetry of $\boldsymbol{\omega}_x^\times$, the following bound can be obtained after some tedious manipulation

$$\|\mathbf{S}\bar{\mathbf{Q}}_i\mathbf{S}\| \le \gamma_2, \quad i \ge 0, \tag{C.8}$$

where γ_2 is given in (68).

Next, from (B.1), one obtains

$$\bar{\mathbf{x}}_{k+j} = \mathbf{S}\breve{\boldsymbol{\Phi}}_{k+j-1,k}\mathbf{S}^{-1}\bar{\mathbf{x}}_k, \quad k,j \ge 0, \tag{C.9}$$

such that

$$\|\mathbf{S}\breve{\boldsymbol{\Phi}}_{k+j-1,k}\mathbf{S}^{-1}\| = \sup_{\bar{\mathbf{x}}_k \ne \mathbf{0}} \frac{\|\bar{\mathbf{x}}_{k+j}\|}{\|\bar{\mathbf{x}}_k\|}, \quad k,j \ge 0. \tag{C.10}$$

From equation (B.11) and (B.12), one obtains for $\bar{\mathbf{x}}_k \ne \mathbf{0}$

$$\frac{\|\bar{\mathbf{x}}_{k+j}\|}{\|\bar{\mathbf{x}}_k\|} \le \left(\frac{\lambda_{\max}(\mathbf{W})}{\lambda_{\min}(\mathbf{W})}\right)^{1/2}\left(1 - \frac{\lambda_{\min}(\mathbf{Z}) - \gamma_1}{\lambda_{\max}(\mathbf{W})}\right)^{j/2}, \tag{C.11}$$

for $k, j \ge 0$. Consequently, from (C.10) this leads to,

$$\begin{aligned}\|\mathbf{S}\breve{\boldsymbol{\Phi}}_{k+j-1,k}\mathbf{S}^{-1}\| \le {} & \left(\frac{\lambda_{\max}(\mathbf{W})}{\lambda_{\min}(\mathbf{W})}\right)^{1/2} \\ & \times \left(1 - \frac{\lambda_{\min}(\mathbf{Z}) - \gamma_1}{\lambda_{\max}(\mathbf{W})}\right)^{j/2},\end{aligned} \tag{C.12}$$

for $k, j \ge 0$. From Theorem 1, $0 \le 1 - (\lambda_{\min}(\mathbf{Z}) - \gamma_1)/\lambda_{\max}(\mathbf{W}) < 1$. Therefore, from (C.12),

$$\lim_{k\to\infty}\|\mathbf{S}\breve{\boldsymbol{\Phi}}_{k,0}\mathbf{S}^{-1}\| = 0. \tag{C.13}$$

Applying equation (C.8) and (C.12) to (C.7) yields

$$\begin{aligned}\|\mathbf{S}\tilde{\mathbf{P}}_k\mathbf{S}\| \le {} & \|\mathbf{S}\breve{\boldsymbol{\Phi}}_{k,0}\mathbf{S}^{-1}\|^2\|\mathbf{S}\tilde{\mathbf{P}}_0\mathbf{S}\| \\ & + \gamma_2\frac{\lambda_{\max}(\mathbf{W})}{\lambda_{\min}(\mathbf{W})}\sum_{i=0}^{k-1}\left(1 - \frac{\lambda_{\min}(\mathbf{Z}) - \gamma_1}{\lambda_{\max}(\mathbf{W})}\right)^i,\end{aligned} \tag{C.14}$$

for $k \ge 1$. Using the well-known identity $\sum_{i=0}^{k-1} a^i = (1-a^k)/(1-a) \le 1/(1-a)$ for all $0 \le a < 1$, setting $a = 1 - (\lambda_{\min}(\mathbf{Z}) - \gamma_1)/\lambda_{\max}(\mathbf{W})$ in (C.14) gives

$$\begin{aligned}\|\mathbf{S}\tilde{\mathbf{P}}_k\mathbf{S}\| \le {} & \|\mathbf{S}\breve{\boldsymbol{\Phi}}_{k,0}\mathbf{S}^{-1}\|^2\|\mathbf{S}\tilde{\mathbf{P}}_0\mathbf{S}\| \\ & + \frac{\lambda_{\max}(\mathbf{W})^2}{\lambda_{\min}(\mathbf{W})}\frac{\gamma_2}{\lambda_{\min}(\mathbf{Z}) - \gamma_1}.\end{aligned} \tag{C.15}$$

Finally, taking the limit superior of both sides of (C.15) and applying (C.13), yields (67).