Observer-Based Adaptive Spacecraft Attitude Control with Guaranteed Performance Bounds

Anton H. J. de Ruiter, *Member, IEEE,*

***Abstract*—This paper considers observer-based adaptive attitude tracking of fully actuated spacecraft with known bounds on the disturbance torques acting on the spacecraft. Given any attitude and angular velocity observer with known ultimate bounds on the estimation errors, sequences of successively less conservative ultimate bounds on the attitude and angular velocity tracking errors are obtained.**

***Index Terms*—Attitude Control, Attitude Observer, Ultimate Boundedness.**

I. Introduction

The conventional approach to designing a spacecraft attitude control system is to initially design a state feedback control law, and then design some state estimation scheme [1] that provides state estimates for use within the control law. As such, a separation principle is assumed, and the controller and state estimation scheme are designed separately.

There has been significant effort on the development of nonlinear observers/filters for attitude estimation, with several providing almost global or global convergence [2]–[10]. However, with a number of these stand-alone observers/filters, there still remains the question regarding their use in a control law, and the existence of an appropriate separation principle. There has also been significant recent effort in the investigation of observer-based attitude control [11]–[16], where stability/convergence results have been obtained. However, in these works the results obtained are specific to the observer under consideration as well as the control law. In addition, for practical implementation an observer-based control scheme needs to be able to accommodate factors such as model uncertainties, disturbance torques and measurement errors. The combination of all three of these factors has received very little attention in the literature with regards to observer-based attitude control.

This paper presents a separation-type principle for observer-based adaptive attitude tracking control of a fully actuated rigid spacecraft, with unknown inertia matrix, disturbance torques and measurement errors. No specific observer is considered. Rather, any observer can be used provided it satisfy some given conditions, namely that the estimates must be continuous, and that the estimation errors must be bounded for all time as well as ultimately bounded with known ultimate bounds (which would result from measurement errors). It is shown that under these conditions, the attitude and angular velocity tracking errors are also ultimately bounded. Furthermore, extending the results from [17], a sequence is presented based on the estimation error ultimate bounds, which provides successively tighter tracking error ultimate bounds. This paper builds on a previous paper [18], where non-adaptive observer-based attitude tracking control is considered.

A. de Ruiter is with the Department of Aerospace Engineering, Ryerson University, Toronto, ON, Canada, M5B 2K3, e-mail: aderuiter@ryerson.ca.

II. Problem Formulation

In this paper, the vector and matrix norms used are $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ and $\|\mathbf{X}\| = \bar{\sigma}(\mathbf{X})$ (where $\bar{\sigma}(\cdot)$ denotes the maximum singular value), respectively. The identity matrix is denoted by $\mathbf{1}$. The unit quaternion is denoted by $(\mathbf{q}, q_4)$, where $\mathbf{q} \in \mathbb{R}^3$ is the vector part of the quaternion, and $q_4 \in \mathbb{R}$ is the scalar part. Associated with any $\mathbf{a} \in \mathbb{R}^3$ is the skew-symmetric matrix $\mathbf{a}^\times \in \mathbb{R}^{3\times3}$ such that for any $\mathbf{b} \in \mathbb{R}^3$, the cross-product between $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a}^\times\mathbf{b}$.

The attitude dynamics of a fully actuated rigid spacecraft are given in body coordinates by [19, p. 59]

$$\dot{\boldsymbol{\omega}} = -\mathbf{P}\boldsymbol{\omega}^\times\mathbf{I}\boldsymbol{\omega} + \mathbf{P}\boldsymbol{\tau}_c + \mathbf{P}\boldsymbol{\tau}_d, \tag{1}$$

where $\mathbf{I}$ is the spacecraft inertia matrix, $\mathbf{P} = \mathbf{I}^{-1}$, $\boldsymbol{\omega}$ is the angular velocity relative to an inertial frame, $\boldsymbol{\tau}_c$ is the control torque and $\boldsymbol{\tau}_d$ is an external disturbance torque.

The desired inertial attitude is denoted $\mathbf{C}_d(t)$ in terms of a rotation matrix. The desired angular velocity $\boldsymbol{\omega}_d(t)$ satisfies $\dot{\mathbf{C}}_d(t) = -\boldsymbol{\omega}_d^\times(t)\mathbf{C}_d(t)$ (see [19, p. 31]). It is assumed that $\boldsymbol{\omega}_d(t)$ and $\dot{\boldsymbol{\omega}}_d(t)$ are continuous and bounded by $\|\boldsymbol{\omega}_d(t)\| \le \bar{w}_d$ and $\|\dot{\boldsymbol{\omega}}_d(t)\| \le \dot{\bar{w}}_d$ for all $t \in \mathbb{R}^+$, respectively. Given the true spacecraft inertial attitude $\mathbf{C}(t)$, the attitude tracking error is defined as

$$\delta\mathbf{C}(t) \triangleq \mathbf{C}(t)\mathbf{C}_d^T(t). \tag{2}$$

When expressed in the true body coordinates, the angular velocity error of the true spacecraft body frame with respect to the desired spacecraft body frame is

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \delta\mathbf{C}\boldsymbol{\omega}_d. \tag{3}$$

Let $(\mathbf{q}, q_4)$ be a quaternion parameterization of the attitude tracking error $\delta\mathbf{C}$, such that [19, p. 30],

$$\delta\mathbf{C}(\mathbf{q}, q_4) = (q_4^2 - \mathbf{q}^T\mathbf{q})\mathbf{1} + 2\mathbf{q}\mathbf{q}^T - 2q_4\mathbf{q}^\times. \tag{4}$$

The quaternion kinematics are given by [19, p. 31]

$$\begin{aligned} \dot{\mathbf{q}} &= \tfrac{1}{2}(\mathbf{q}^\times + q_4\mathbf{1})\delta\boldsymbol{\omega}, \\ \dot{q}_4 &= -\tfrac{1}{2}\mathbf{q}^T\delta\boldsymbol{\omega}. \end{aligned} \tag{5}$$

It is assumed that the disturbance torque has a known bound,

$$\|\boldsymbol{\tau}_d(t)\| \le \bar{\tau}_d, \ \ \forall t \in \mathbb{R}^+. \tag{6}$$

It is further assumed that an observer provides estimates of the attitude and body-rate given by

$$\hat{\mathbf{C}} = \mathbf{C}(\mathbf{v}_q, v_{q4})\mathbf{C} \tag{7}$$

and

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega} + \mathbf{v}_\omega, \tag{8}$$

respectively, where $(\mathbf{v}_q, v_{q4})$ is a quaternion representation of the multiplicative attitude estimation error, and $\mathbf{v}_\omega$ is the body-rate estimation error. An estimate $(\hat{\mathbf{q}}, \hat{q}_4)$ of the tracking error $(\mathbf{q}, q_4)$ may now be obtained from

$$\delta\hat{\mathbf{C}}(\hat{\mathbf{q}}, \hat{q}_4) = \hat{\mathbf{C}}\mathbf{C}_d^T = \mathbf{C}(\mathbf{v}_q, v_{q4})\delta\mathbf{C}(\mathbf{q}, q_4),$$

which leads to [19, p. 17]

$$\begin{aligned} \hat{\mathbf{q}} &= \mathbf{q} + \left((v_{q4} - 1)\mathbf{q} + q_4\mathbf{v}_q + \mathbf{q}^\times\mathbf{v}_q\right), \\ \hat{q}_4 &= q_4 + \left((v_{q4} - 1)q_4 - \mathbf{q}^T\mathbf{v}_q\right). \end{aligned} \tag{9}$$

Assumption 1

1) *$\hat{\mathbf{q}}(t), \hat{q}_4(t), \hat{\boldsymbol{\omega}}(t)$ are continuous.*
2) *The estimation errors $\mathbf{v}_q(t), v_{q4}(t), \mathbf{v}_\omega(t)$ are differentiable, and $\mathbf{v}_\omega(t)$, $\dot{v}_{q4}(t)$, $\dot{\mathbf{v}}_q(t)$ and $\dot{\mathbf{v}}_\omega(t)$ are bounded for all $t \in \mathbb{R}^+$.*
3) *There exist $0 \le \bar{v}_q < 1$, $0 \le \bar{v}_\omega$, $0 \le \dot{\bar{v}}_q$, and $0 \le \dot{\bar{v}}_\omega$ such that*

$$\limsup_{t\to\infty} \|\mathbf{v}_q(t)\| \le \bar{v}_q,$$
$$\limsup_{t\to\infty} \|\mathbf{v}_\omega(t)\| \le \bar{v}_\omega,$$
$$\limsup_{t\to\infty} \|\dot{\mathbf{v}}_q(t)\| \le \dot{\bar{v}}_q,$$
$$\limsup_{t\to\infty} \|\dot{\mathbf{v}}_\omega(t)\| \le \dot{\bar{v}}_\omega,$$

Using the quaternion unit length constraint, one also obtains

$$\limsup_{t\to\infty} |\dot{v}_{q4}(t)| \le \frac{\bar{v}_q\dot{\bar{v}}_q}{\sqrt{1-\bar{v}_q^2}} \triangleq \dot{\bar{v}}_{q4}. \tag{10}$$

Remark 1 It is assumed without loss of generality, that given the ultimate bound on $\mathbf{v}_q(t)$ in property 3), the scalar part of the estimation error $v_{q4}(t)$ eventually becomes positive. This is justified, because both $(\mathbf{q}, q_4)$ and $(-\mathbf{q}, -q_4)$ represent the same attitude [19]. Therefore, in (9) one of the choices of the quaternion will ensure that $v_{q4}(t)$ eventually becomes positive. By continuity of both $(\hat{\mathbf{q}}(t), \hat{q}_4(t))$, and $(\mathbf{q}(t), q_4(t))$, this choice will never change for a given trajectory of the closed-loop system (to be defined shortly). Note that in this paper, no consideration of (or prevention of) the unwinding phenomenon [21] is given. The possibility of unwinding is made as a trade-off for global convergence results in $\mathbf{q}$ (the vector part of the tracking error quaternion) and $\delta\boldsymbol{\omega}$.

III. Adaptive Control with State Feedback

Define an auxiliary desired angular velocity as

$$\bar{\boldsymbol{\omega}}_d \triangleq \delta\mathbf{C}(\mathbf{q}, q_4)\boldsymbol{\omega}_d - \lambda\mathbf{q}, \tag{11}$$

with $\lambda > 0$ some constant positive gain. This has derivative

$$\begin{aligned} \dot{\bar{\boldsymbol{\omega}}}_d &= -\delta\boldsymbol{\omega}^\times\delta\mathbf{C}(\mathbf{q}, q_4)\boldsymbol{\omega}_d + \delta\mathbf{C}(\mathbf{q}, q_4)\dot{\boldsymbol{\omega}}_d \\ &\quad -\frac{\lambda}{2}(\mathbf{q}^\times + q_4\mathbf{1})\delta\boldsymbol{\omega}. \end{aligned} \tag{12}$$

Define a filtered angular velocity error as

$$\tilde{\boldsymbol{\omega}} \triangleq \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_d = \delta\boldsymbol{\omega} + \lambda\mathbf{q}. \tag{13}$$

By direct expansion, there exists a $\boldsymbol{\theta} \in \mathbb{R}^{18}$ (a function of $\mathbf{P}$ and $\mathbf{I}$) such that

$$\boldsymbol{\Phi}(\boldsymbol{\omega})\boldsymbol{\theta} = \mathbf{P}\boldsymbol{\omega}^\times\mathbf{I}\boldsymbol{\omega}, \tag{14}$$

where

$$\boldsymbol{\Phi}(\boldsymbol{\omega}) = \text{diag}\left\{\boldsymbol{\phi}(\boldsymbol{\omega}), \boldsymbol{\phi}(\boldsymbol{\omega}), \boldsymbol{\phi}(\boldsymbol{\omega})\right\}, \tag{15}$$

and

$$\boldsymbol{\phi}(\boldsymbol{\omega}) = \left[\begin{array}{cccccc} \omega_x^2 & \omega_y^2 & \omega_z^2 & \omega_x\omega_y & \omega_x\omega_z & \omega_y\omega_z \end{array}\right]. \tag{16}$$

Therefore, the attitude dynamics (1) become

$$\dot{\boldsymbol{\omega}} = -\boldsymbol{\Phi}(\boldsymbol{\omega})\boldsymbol{\theta} + \mathbf{P}\boldsymbol{\tau}_c + \boldsymbol{\tau}_{dP}. \tag{17}$$

The quantities $\boldsymbol{\theta}$ and $\mathbf{P}$ are unknown, but constant. Their estimates are denoted by $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{P}}$, respectively. Noting that $\mathbf{P}$ is symmetric, it can be shown that

$$\mathbf{P}\boldsymbol{\tau}_c = \boldsymbol{\Psi}(\boldsymbol{\tau}_c)\boldsymbol{\theta}_p \tag{18}$$

where

$$\boldsymbol{\Psi}(\boldsymbol{\tau}_c) = \left[\begin{array}{cccccc} \tau_{cx} & 0 & 0 & \tau_{cy} & \tau_{cz} & 0 \\ 0 & \tau_{cy} & 0 & \tau_{cx} & 0 & \tau_{cz} \\ 0 & 0 & \tau_{cz} & 0 & \tau_{cx} & \tau_{cy} \end{array}\right],$$

$$\boldsymbol{\theta}_p = \left[\begin{array}{cccccc} P_{11} & P_{22} & P_{33} & P_{12} & P_{13} & P_{23} \end{array}\right]^T,$$

and P_{ij} denotes the ij^{th} term of $\mathbf{P}$. As such, the attitude dynamics (1) may be rewritten as

$$\dot{\boldsymbol{\omega}} = -\boldsymbol{\Phi}(\boldsymbol{\omega})\boldsymbol{\theta} + \hat{\mathbf{P}}\boldsymbol{\tau}_c - \boldsymbol{\Psi}(\boldsymbol{\tau}_c)\tilde{\boldsymbol{\theta}}_p + \boldsymbol{\tau}_{dP}, \tag{19}$$

where $\tilde{\boldsymbol{\theta}}_p = \hat{\boldsymbol{\theta}}_p - \boldsymbol{\theta}_p$ and $\hat{\boldsymbol{\theta}}_p$ denotes the estimate of $\boldsymbol{\theta}_p$.

It is assumed that

$$\begin{aligned} \boldsymbol{\theta} &\in F \triangleq \left\{\mathbf{v} \in \mathbb{R}^{18} : \|\mathbf{v} - \boldsymbol{\theta}^*\| \le \eta\right\}, \\ \boldsymbol{\theta}_p &\in F_p \triangleq \left\{\mathbf{v} \in \mathbb{R}^{6} : \|\mathbf{v} - \boldsymbol{\theta}_p^*\| \le \eta_p\right\}, \end{aligned}$$

where $\boldsymbol{\theta}^* \in \mathbb{R}^{18}$, $\boldsymbol{\theta}_p^* \in \mathbb{R}^6$, $\eta > 0$ and $\eta_p > 0$ are known parameters. A known bound $\underline{\lambda}_I \le \lambda_{min}(\mathbf{I})$ is also assumed.

Consider the control and adaptation laws

$$\boldsymbol{\tau}_c = \hat{\mathbf{P}}^{-1}\left[\dot{\bar{\boldsymbol{\omega}}}_d - k\tilde{\boldsymbol{\omega}} + \boldsymbol{\Phi}(\boldsymbol{\omega})\hat{\boldsymbol{\theta}}\right], \tag{20}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \gamma \text{Proj}_1\left(\hat{\boldsymbol{\theta}}, -\boldsymbol{\Phi}(\boldsymbol{\omega})^T\tilde{\boldsymbol{\omega}}\right), \;\; \hat{\boldsymbol{\theta}}(0) = \boldsymbol{\theta}^*, \tag{21}$$

$$\dot{\hat{\boldsymbol{\theta}}}_p = \gamma_p \text{Proj}_2\left(\hat{\boldsymbol{\theta}}_p, \boldsymbol{\Psi}(\boldsymbol{\tau}_c)^T\tilde{\boldsymbol{\omega}}\right), \;\; \hat{\boldsymbol{\theta}}_p(0) = \boldsymbol{\theta}_p^*, \tag{22}$$

where $k, \gamma, \gamma_p > 0$ are scalar positive gains. Reference [17] provides conditions to guarantee that $\hat{\mathbf{P}}$ remains invertible for all $t \in \mathbb{R}^+$.

The projection operators in (21) and (22) are defined as [22]

$$\text{Proj}_i(\boldsymbol{\upsilon}, \mathbf{y}) = \begin{cases} \mathbf{y} - \mathbf{f}_i(\boldsymbol{\upsilon}, \mathbf{y}), & \min\{d_i(\boldsymbol{\upsilon}), \nabla d_i(\boldsymbol{\upsilon})^T\mathbf{y}\} > 0, \\ \mathbf{y}, & \text{otherwise}, \end{cases} \tag{23}$$

for $i = 1, 2$, where

$$\mathbf{f}_i(\boldsymbol{\upsilon}, \mathbf{y}) = \frac{d_i(\boldsymbol{\upsilon})\nabla d_i(\boldsymbol{\upsilon})\nabla d_i(\boldsymbol{\upsilon})^T\mathbf{y}}{||\nabla d_i(\boldsymbol{\upsilon})||^2}, \tag{24}$$

$$d_1(\boldsymbol{\upsilon}) = \frac{||\boldsymbol{\upsilon} - \boldsymbol{\theta}^*||^2 - \eta^2}{\epsilon_{\upsilon 1}\eta^2}, \;\; \epsilon_{\upsilon 1} > 0, \tag{25}$$

$$d_2(\boldsymbol{\upsilon}) = \frac{||\boldsymbol{\upsilon} - \boldsymbol{\theta}_p^*||^2 - \eta_p^2}{\epsilon_{\upsilon 2}\eta_p^2}, \;\; \epsilon_{\upsilon 2} > 0. \tag{26}$$

It can be shown that the tracking errors $(\mathbf{q}, \delta\boldsymbol{\omega})$ resulting from the application of (20), (21) and (22) are globally convergent in the absence of disturbance torques.

The main result in this paper is a steady-state performance bound for the tracking error if the control and adaptation laws in (20), (21) and (22) are formed using state estimates in place of the true states, in the presence of disturbance torques.

IV. Main Result

For observer-based control the control and adaptation laws in (20), (21) and (22) become

$$\boldsymbol{\tau}_c = \hat{\mathbf{P}}^{-1}\left[\dot{\bar{\boldsymbol{\omega}}}_d^{\wedge} - k\tilde{\boldsymbol{\omega}}^{\wedge} + \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}})\hat{\boldsymbol{\theta}}\right], \tag{27}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \begin{cases} \gamma \text{Proj}_1\left(\hat{\boldsymbol{\theta}}, -\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}})^T\tilde{\boldsymbol{\omega}}^{\wedge}\right), & \|\tilde{\boldsymbol{\omega}}^{\wedge}\| > \Omega_{\wedge}, \\ \mathbf{0}, & \|\tilde{\boldsymbol{\omega}}^{\wedge}\| \le \Omega_{\wedge}, \end{cases}, \tag{28}$$

$$\dot{\hat{\boldsymbol{\theta}}}_p = \begin{cases} \gamma_p \text{Proj}_2\left(\hat{\boldsymbol{\theta}}_p, \boldsymbol{\Psi}(\boldsymbol{\tau}_c)^T\tilde{\boldsymbol{\omega}}^{\wedge}\right), & \|\tilde{\boldsymbol{\omega}}^{\wedge}\| > \Omega_{\wedge}, \\ \mathbf{0}, & \|\tilde{\boldsymbol{\omega}}^{\wedge}\| \le \Omega_{\wedge}, \end{cases} \tag{29}$$

and

$$\begin{aligned} &\bar{\boldsymbol{\omega}}_d^{\wedge} = \delta\hat{\mathbf{C}}\boldsymbol{\omega}_d - \lambda\hat{\mathbf{q}}, \; \tilde{\boldsymbol{\omega}}^{\wedge} = \hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}_d^{\wedge}, \; \delta\boldsymbol{\omega}^{\wedge} = \hat{\boldsymbol{\omega}} - \delta\hat{\mathbf{C}}\boldsymbol{\omega}_d, \\ &\dot{\bar{\boldsymbol{\omega}}}_d^{\wedge} = -\delta\boldsymbol{\omega}^{\wedge\times}\delta\hat{\mathbf{C}}\boldsymbol{\omega}_d + \delta\hat{\mathbf{C}}\dot{\boldsymbol{\omega}}_d - \tfrac{\lambda}{2}\left(\hat{\mathbf{q}}^{\times} + \hat{q}_4\mathbf{1}\right)\delta\boldsymbol{\omega}^{\wedge}, \end{aligned} \tag{30}$$

with initial conditions $\hat{\boldsymbol{\theta}}(0) = \boldsymbol{\theta}^*$ and $\hat{\boldsymbol{\theta}}_p(0) = \boldsymbol{\theta}_p^*$, where $k, \gamma, \gamma_p > 0$ are scalar positive gains, and $\Omega_{\wedge} > 0$ is a deadzone size. Note that the deadzone has been added to the adaptation laws (28) and (29) (compare to (21) and (22)), in order to accommodate observer errors and disturbance torques. Using equations (3), (11), (12), (13), (8) and (9), the quantities in (30) may be written as

$$\begin{aligned} \bar{\boldsymbol{\omega}}_d^{\wedge} &= \bar{\boldsymbol{\omega}}_d + \mathbf{v}_{\bar{\omega}_d}, \\ \tilde{\boldsymbol{\omega}}^{\wedge} &= \tilde{\boldsymbol{\omega}} + \mathbf{v}_{\tilde{\omega}}, \\ \delta\boldsymbol{\omega}^{\wedge} &= \delta\boldsymbol{\omega} + \mathbf{v}_{\delta\omega}, \\ \dot{\bar{\boldsymbol{\omega}}}_d^{\wedge} &= \dot{\bar{\boldsymbol{\omega}}}_d + \mathbf{v}_{\dot{\bar{\omega}}_d}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mathbf{v}_{\bar{\omega}_d} &= 2\mathbf{v}_q^{\times}\left[\mathbf{v}_q^{\times} - v_{q4}\mathbf{1}\right]\delta\mathbf{C}\boldsymbol{\omega}_d \\ &\quad -\lambda\left[(v_{q4} - 1)\mathbf{q} + (q_4\mathbf{1} + \mathbf{q}^{\times})\mathbf{v}_q\right], \\ \mathbf{v}_{\tilde{\omega}} &= \mathbf{v}_{\omega} - \mathbf{v}_{\bar{\omega}_d}, \\ \mathbf{v}_{\delta\omega} &= \mathbf{v}_{\omega} - 2\mathbf{v}_q^{\times}\left[\mathbf{v}_q^{\times} - v_{q4}\mathbf{1}\right]\delta\mathbf{C}\boldsymbol{\omega}_d, \\ \mathbf{v}_{\dot{\bar{\omega}}_d} &= -\mathbf{v}_{\delta\omega}^{\times}\delta\hat{\mathbf{C}}\boldsymbol{\omega}_d - 2\delta\boldsymbol{\omega}^{\times}\mathbf{v}_q^{\times}\left[\mathbf{v}_q^{\times} - v_{q4}\mathbf{1}\right]\delta\mathbf{C}\boldsymbol{\omega}_d \\ &\quad +2\mathbf{v}_q^{\times}\left[\mathbf{v}_q^{\times} - v_{q4}\mathbf{1}\right]\delta\mathbf{C}\dot{\boldsymbol{\omega}}_d \\ &\quad -\tfrac{\lambda}{2}(v_{q4} - 1)\left[\mathbf{q}^{\times} + q_4\mathbf{1}\right]\delta\boldsymbol{\omega} - \tfrac{\lambda}{2}\left[\hat{\mathbf{q}}^{\times} + \hat{q}_4\mathbf{1}\right]\mathbf{v}_{\delta\omega} \\ &\quad -\tfrac{\lambda}{2}\left[(q_4\mathbf{1} + \mathbf{q}^{\times})\mathbf{v}_q^{\times} - \mathbf{q}\mathbf{v}_q^T\right]\delta\boldsymbol{\omega}. \end{aligned} \tag{32}$$

The quantities in (32) can be bounded by

$$\|\mathbf{v}_{\bar{\omega}_d}\| \le \bar{v}_{\bar{\omega}_d}, \; \|\mathbf{v}_{\tilde{\omega}}\| \le \bar{v}_{\tilde{\omega}}, \; \|\mathbf{v}_{\delta\omega}\| \le \bar{v}_{\delta\omega}, \; \|\mathbf{v}_{\dot{\bar{\omega}}_d}\| \le \bar{v}_{\dot{\bar{\omega}}_d}, \tag{33}$$

where

$$\begin{aligned} \bar{v}_{\bar{\omega}_d} &= \lambda\|\mathbf{q}\||v_{q4} - 1| + (2\bar{w}_d + \lambda)\|\mathbf{v}_q\|, \\ \bar{v}_{\tilde{\omega}} &= \|\mathbf{v}_{\omega}\| + \bar{v}_{\bar{\omega}_d}, \\ \bar{v}_{\delta\omega} &= \|\mathbf{v}_{\omega}\| + 2\|\mathbf{v}_q\|\bar{w}_d, \\ \bar{v}_{\dot{\bar{\omega}}_d} &= \left[\|\tilde{\boldsymbol{\omega}}\| + \lambda\|\mathbf{q}\|\right]\big[2\|\mathbf{v}_q\|\bar{w}_d \\ &\quad +\tfrac{\lambda}{2}\left(|v_{q4} - 1| + \|\mathbf{v}_q\|(1 + \|\mathbf{q}\|)\right)\big] \\ &\quad +\bar{v}_{\delta\omega}\left[\bar{w}_d + \tfrac{\lambda}{2}\right] + 2\|\mathbf{v}_q\|\dot{\bar{w}}_d. \end{aligned} \tag{34}$$

The projection operator has two important properties [22]:

Projection Operator Properties

1) $$\left(\boldsymbol{\upsilon} - \bar{\boldsymbol{\theta}}\right)^T\left(\mathit{Proj}_1(\boldsymbol{\upsilon}, \mathbf{y}) - \mathbf{y}\right) \le \mathbf{0},$$

for any $\boldsymbol{\upsilon} \in \mathbb{R}^{18}$, $\mathbf{y} \in \mathbb{R}^{18}$ *and any* $\bar{\boldsymbol{\theta}} \in F$, *and*

$$\left(\boldsymbol{\upsilon}_p - \bar{\boldsymbol{\theta}}_p\right)^T\left(\mathit{Proj}_2(\boldsymbol{\upsilon}_p, \mathbf{y}_p) - \mathbf{y}_p\right) \le \mathbf{0},$$

for any $\boldsymbol{\upsilon}_p \in \mathbb{R}^6$, $\mathbf{y}_p \in \mathbb{R}^6$ *and any* $\bar{\boldsymbol{\theta}}_p \in F_p$.

2) *If*

$$\hat{\boldsymbol{\theta}}(0) \in G \triangleq \left\{\mathbf{v} \in \mathbb{R}^{18} : \|\mathbf{v} - \boldsymbol{\theta}^*\| \le \sqrt{1 + \epsilon_{\upsilon 1}}\eta\right\},$$

and

$$\hat{\boldsymbol{\theta}}_p(0) \in G_p \triangleq \left\{\mathbf{v} \in \mathbb{R}^{6} : \|\mathbf{v} - \boldsymbol{\theta}_p^*\| \le \sqrt{1 + \epsilon_{\upsilon 2}}\eta_p\right\},$$

then with the adaptation laws (28) and (29), $\hat{\boldsymbol{\theta}}(t) \in G$ *and* $\hat{\boldsymbol{\theta}}_p(t) \in G_p$ *for all* $t \in \mathbb{R}^+$.

Making use of (31), and substituting (27) into (19), the closed-loop equation for the estimated filtered error is

$$\begin{aligned}\dot{\tilde{\boldsymbol{\omega}}}^{\wedge} &= \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}})\tilde{\boldsymbol{\theta}} - k\tilde{\boldsymbol{\omega}}^{\wedge} - \boldsymbol{\Psi}(\boldsymbol{\tau}_c)\tilde{\boldsymbol{\theta}}_p \\ &\quad + [\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}) - \boldsymbol{\Phi}(\boldsymbol{\omega})]\,\boldsymbol{\theta} + \mathbf{v}_{\dot{\tilde{\omega}}_d} + \dot{\mathbf{v}}_{\tilde{\omega}} + \boldsymbol{\tau}_{dP}.\end{aligned} \quad (35)$$

Bounds are now given for some of the terms on the right-hand side of (35). To this end, define the quantities

$$\begin{aligned}
a &= \sqrt{6}\|\mathbf{v}_\omega\|\,[\|\boldsymbol{\theta}^*\| + \eta]\,, \\
b &= \left[\|\mathbf{v}_\omega\|^2 + \sqrt{6}\|\mathbf{v}_\omega\|\,(\bar{v}_{\tilde{\omega}} + \bar{w}_d + \lambda\|\mathbf{q}\|)\right] \\
&\quad \times [\|\boldsymbol{\theta}^*\| + \eta]\,, \\
c &= 2\|\mathbf{v}_q\|\bar{w}_d + \frac{\lambda}{2}\left(|v_{q4} - 1| + \|\mathbf{v}_q\|(1 + \|\mathbf{q}\|)\right), \\
d &= \left[2\|\mathbf{v}_q\|\bar{w}_d + \frac{\lambda}{2}\left(|v_{q4} - 1| + \|\mathbf{v}_q\|(1 + \|\mathbf{q}\|)\right)\right] \\
&\quad \times [\bar{v}_{\tilde{\omega}} + \lambda\|\mathbf{q}\|] \\
&\quad + \bar{v}_{\delta\omega}\left[\bar{w}_d + \frac{\lambda}{2}\right] + 2\|\mathbf{v}_q\|\dot{\bar{w}}_d, \\
e &= 2\|\mathbf{v}_q\|\bar{w}_d + \frac{\lambda}{2}\left[|v_{q4} - 1| + \|\mathbf{v}_q\|(1 + \|\mathbf{q}\|)\right], \\
f &= \left[2\|\mathbf{v}_q\|\bar{w}_d + \frac{\lambda}{2}\left[|v_{q4} - 1| + \|\mathbf{v}_q\|(1 + \|\mathbf{q}\|)\right]\right] \\
&\quad \times [\bar{v}_{\tilde{\omega}} + \lambda\|\mathbf{q}\|] \\
&\quad + \|\dot{\mathbf{v}}_\omega\| + 2\left[\|\dot{\mathbf{v}}_q\|(\|\mathbf{v}_q\| + 1) + \|\mathbf{v}_q\||\dot{v}_{q4}|\right]\bar{w}_d \\
&\quad + 2\|\mathbf{v}_q\|\dot{\bar{w}}_d + \lambda\left[|\dot{v}_{q4}|\|\mathbf{q}\| + \|\dot{\mathbf{v}}_q\|\right],
\end{aligned} \quad (36)$$

where $\bar{v}_{\delta\omega}, \bar{v}_{\tilde{\omega}}$ are given in (34). After some work, the following bounds can be obtained [17]

$$\|\,[\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}) - \boldsymbol{\Phi}(\boldsymbol{\omega})]\,\boldsymbol{\theta}\| \le a\|\tilde{\boldsymbol{\omega}}^{\wedge}\| + b, \quad (37)$$

$$\|\mathbf{v}_{\dot{\tilde{\omega}}_d}\| \le c\|\tilde{\boldsymbol{\omega}}^{\wedge}\| + d, \quad (38)$$

$$\|\dot{\mathbf{v}}_{\tilde{\omega}}\| \le e\|\tilde{\boldsymbol{\omega}}^{\wedge}\| + f. \quad (39)$$

The deadzone will be sized based upon the ultimate error bounds on the estimation errors and their rates given in Assumption 1. Accordingly, the following quantities are defined

$$\hat{\bar{v}}_{\tilde{\omega}} = \bar{v}_\omega + \lambda\left[1 - \sqrt{1 - \bar{v}_q^2}\right] + [2\bar{w}_d + \lambda]\,\bar{v}_q, \quad (40)$$

$$\hat{a}(\bar{v}_\omega) = \sqrt{6}\bar{v}_\omega\,[\|\boldsymbol{\theta}^*\| + \eta]\,, \quad (41)$$

$$\hat{b}(\bar{v}_q, \bar{v}_\omega) = \left[\bar{v}_\omega^2 + \sqrt{6}\bar{v}_\omega\left(\hat{\bar{v}}_{\tilde{\omega}} + \bar{w}_d + \lambda\right)\right][\|\boldsymbol{\theta}^*\| + \eta]\,, \quad (42)$$

$$\hat{c}(\bar{v}_q) = 2\bar{v}_q\bar{w}_d + \frac{\lambda}{2}\left(1 - \sqrt{1 - \bar{v}_q^2} + 2\bar{v}_q\right), \quad (43)$$

$$\begin{aligned}&\hat{d}(\bar{v}_q, \bar{v}_\omega) = \\ &\left[2\bar{v}_q\bar{w}_d + \tfrac{\lambda}{2}\left(1 - \sqrt{1 - \bar{v}_q^2} + 2\bar{v}_q\right)\right]\left[\hat{\bar{v}}_{\tilde{\omega}} + \lambda\right] \\ &+ 2\bar{v}_q\dot{\bar{w}}_d + [\bar{v}_\omega + 2\bar{v}_q\bar{w}_d]\left[\bar{w}_d + \tfrac{\lambda}{2}\right],\end{aligned} \quad (44)$$

$$\hat{e}(\bar{v}_q) = 2\bar{v}_q\bar{w}_d + \frac{\lambda}{2}\left[1 - \sqrt{1 - \bar{v}_q^2} + 2\bar{v}_q\right], \quad (45)$$

$$\begin{aligned}&\hat{f}(\bar{v}_q, \bar{v}_\omega, \dot{\bar{v}}_q, \dot{\bar{v}}_\omega) = \\ &\left[2\bar{v}_q\bar{w}_d + \tfrac{\lambda}{2}\left[1 - \sqrt{1 - \bar{v}_q^2} + 2\bar{v}_q\right]\right]\left[\hat{\bar{v}}_{\tilde{\omega}} + \lambda\right] \\ &+ \dot{\bar{v}}_\omega + 2\left[\dot{\bar{v}}_q(\bar{v}_q + 1) + \bar{v}_q\dot{\bar{v}}_{q4}\right]\bar{w}_d \\ &+ 2\bar{v}_q\dot{\bar{w}}_d + \lambda\left[\dot{\bar{v}}_{q4} + \dot{\bar{v}}_q\right].\end{aligned} \quad (46)$$

One can note that the quantities in (41) to (46) are simply upper bounds on the quantities in (36) where $\|\mathbf{q}\|$ has been upper-bounded by one, and the estimation errors and their rates have been upper-bounded by their ultimate bounds given in Assumption 1.
The deadzone size is chosen to satisfy

$$\Omega_\wedge > \frac{\hat{b}(\bar{v}_q, \bar{v}_\omega) + \hat{d}(\bar{v}_q, \bar{v}_\omega) + \hat{f}(\bar{v}_q, \bar{v}_\omega, \dot{\bar{v}}_q, \dot{\bar{v}}_\omega) + \bar{\tau}_d/\underline{\lambda}_I}{k - (\hat{a}(\bar{v}_\omega) + \hat{c}(\bar{v}_q) + \hat{e}(\bar{v}_q))} \quad (47)$$

An iterative algorithm is now presented for finding successively tighter ultimate upper bounds on $\|\mathbf{q}(t)\|$ and $\|\delta\boldsymbol{\omega}(t)\|$ for the closed-loop system.

Algorithm 1
Set $\bar{q}_0 = 1$. For $i = 1, ..., n$, where n is some (user determined) finite positive integer, compute

$$\begin{aligned}\bar{v}_{\tilde{\omega}}^i &= \bar{v}_\omega + \bar{q}_{i-1}\lambda\left[1 - \sqrt{1 - \bar{v}_q^2}\right] + [2\bar{w}_d + \lambda]\,\bar{v}_q, \\ \bar{r}_i &= \Omega_\wedge + \bar{v}_{\tilde{\omega}}^i, \\ \bar{q}_i &= \tfrac{\bar{r}_i}{\lambda}.\end{aligned} \quad (48)$$

Before presenting the main result on observer-based adaptive attitude control, a preliminary result is needed for the case where the ultimate bounds in Assumption 1 are in fact uniform bounds. To this end, the following result is adapted from [17].

Proposition 1 [17]
Assume that $0 \le \bar{q}_1 < 1$, $k > \hat{a}(\bar{v}_\omega) + \hat{c}(\bar{v}_q) + \hat{e}(\bar{v}_q)$ and that the deadzone satisfies (47). Then, the sequences $\{\bar{r}_i\}$ and $\{\bar{q}_i\}$ in Algorithm 1 are decreasing and convergent. Furthermore, suppose that the estimation errors and their rates satisfy $\|\mathbf{v}_q(t)\| \le \bar{v}_q < 1$, $\|\mathbf{v}_\omega(t)\| \le \bar{v}_\omega$, $\|\dot{\mathbf{v}}_q(t)\| \le \dot{\bar{v}}_q$, and $\|\dot{\mathbf{v}}_\omega(t)\| \le \dot{\bar{v}}_\omega$ for all $t \in \mathbb{R}^+$. Then, the control and adaptation laws (27), (28) and (29) applied to the system given by (1), (3) and (5), yield

$$\begin{aligned}\limsup_{t\to\infty}\|\mathbf{q}(t)\| &\le \bar{q}_i, \\ \limsup_{t\to\infty}\|\delta\boldsymbol{\omega}(t)\| &\le 2\bar{r}_i,\end{aligned}$$

for $i \ge 1$. Furthermore, let $\bar{q} = \lim_{i\to\infty}\bar{q}_i$ and $\bar{r} = \lim_{i\to\infty}\bar{r}_i$. Then,

$$\begin{aligned}\limsup_{t\to\infty}\|\mathbf{q}(t)\| &\le \bar{q}, \\ \limsup_{t\to\infty}\|\delta\boldsymbol{\omega}(t)\| &\le 2\bar{r}.\end{aligned}$$

The main result is now presented.

Theorem 1 *Suppose that an observer satisfies Assumption 1. Consider the sequence defined by*

Algorithm 1, computed using the ultimate observer bounds in Assumption 1. Assume that $0 \le \bar{q}_1 < 1$, $k > \hat{a}(\bar{v}_\omega) + \hat{c}(\bar{v}_q) + \hat{e}(\bar{v}_q)$ *and that the deadzone satisfies (47). Then, the control and adaptation laws (27), (28) and (29) applied to the system given by (1), (3) and (5), yield*

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\le \bar{q}_i, \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\le 2\bar{r}_i, \end{aligned}$$

for $i \ge 1$. *Furthermore, let* $\bar{q} = \lim_{i\to\infty} \bar{q}_i$ *and* $\bar{r} = \lim_{i\to\infty} \bar{r}_i$. *Then,*

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\le \bar{q}, \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\le 2\bar{r}. \end{aligned}$$

Proof It is first shown that all signals remain finite for all time. By "Projection Operator Properties" property 2), the parameter estimates $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}_p$ cannot leave the compact sets G and G_p respectively. Since the true parameters $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_p$ are finite, the parameter estimation errors $\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}_p$ must remain finite. It remains to show that the estimated filtered error $\tilde{\boldsymbol{\omega}}^\wedge$ remains finite for all time. Since the deadzone $\{\tilde{\boldsymbol{\omega}}^\wedge : \|\tilde{\boldsymbol{\omega}}^\wedge\| \le \Omega_\wedge\}$ is a compact set, it suffices to examine the worst case behavior of $\tilde{\boldsymbol{\omega}}^\wedge$ outside the deadzone. To this end, consider the Lyapunov-like function

$$V(\tilde{\boldsymbol{\omega}}^\wedge, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}_p) = \frac{1}{2}\tilde{\boldsymbol{\omega}}^{\wedge T}\tilde{\boldsymbol{\omega}}^\wedge + \frac{1}{2\gamma}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} + \frac{1}{2\gamma_p}\tilde{\boldsymbol{\theta}}_p^T\tilde{\boldsymbol{\theta}}_p. \quad (49)$$

Differentiation of (49) along a closed-loop trajectory of (35), applying "Projection Operator Properties" property 1), and making use of the bounds given in (37), (38) and (39), one eventually obtains

$$\dot{V} \le (a + c + e - k)\|\tilde{\boldsymbol{\omega}}^\wedge\|^2 + (b + d + f + \bar{\tau}_d/\underline{\lambda}_I)\|\tilde{\boldsymbol{\omega}}^\wedge\|. \quad (50)$$

Application of Young's inequality yields

$$\dot{V} \le (a+c+e+1/2-k)\|\tilde{\boldsymbol{\omega}}^\wedge\|^2 + (b+d+f+\bar{\tau}_d/\underline{\lambda}_I)^2/2. \quad (51)$$

By Assumption 1, the estimation errors and their rates are all bounded, hence the quantities a to f in (36) can also be bounded. Therefore, from (51) there exist finite $g, h > 0$ such that

$$\dot{V} \le g\|\tilde{\boldsymbol{\omega}}^\wedge\|^2/2 + h. \quad (52)$$

Finally, noting that $\|\tilde{\boldsymbol{\omega}}^\wedge\|^2 \le 2V$, one obtains

$$\dot{V} \le gV + h. \quad (53)$$

Application of the comparison principle [20] leads to

$$\bar{V}(t) \le e^{gt}(\bar{V}(0) + h/g) - h/g,$$

where $\bar{V}(t) \stackrel{\Delta}{=} V(\tilde{\boldsymbol{\omega}}^\wedge(t), \tilde{\boldsymbol{\theta}}(t), \tilde{\boldsymbol{\theta}}_p(t))$. Consequently, the estimated filtered error $\tilde{\boldsymbol{\omega}}^\wedge$ remains finite. By (31), (33) and (34), the true filtered error $\tilde{\boldsymbol{\omega}}$ remains finite also, and consequently by (13) so does the angular velocity $\boldsymbol{\omega}(t)$, and therefore all other signals in the closed-loop.

By Assumption 1, for any $\epsilon > 0$, there exists a time $T \ge 0$ such that

$$\begin{aligned} \|\mathbf{v}_q(t)\| &\le \bar{v}_q + \epsilon \stackrel{\Delta}{=} \bar{v}_{q\epsilon}, \\ \|\mathbf{v}_\omega(t)\| &\le \bar{v}_\omega + \epsilon \stackrel{\Delta}{=} \bar{v}_{\omega\epsilon}, \\ \|\dot{\mathbf{v}}_q(t)\| &\le \dot{\bar{v}}_q + \epsilon \stackrel{\Delta}{=} \dot{\bar{v}}_{q\epsilon}, \\ \|\dot{\mathbf{v}}_\omega(t)\| &\le \dot{\bar{v}}_\omega + \epsilon \stackrel{\Delta}{=} \dot{\bar{v}}_{\omega\epsilon}, \end{aligned}$$

for all $t \ge T$. In particular, it can be seen from (41) to (47), that for small enough $\epsilon > 0$, $k > \hat{a}(\bar{v}_{\omega\epsilon}) + \hat{c}(\bar{v}_{q\epsilon}) + \hat{e}(\bar{v}_{q\epsilon})$ and

$$\Omega_\wedge > \frac{\hat{b}(\bar{v}_{q\epsilon}, \bar{v}_{\omega\epsilon}) + \hat{d}(\bar{v}_{q\epsilon}, \bar{v}_{\omega\epsilon}) + \hat{f}(\bar{v}_{q\epsilon}, \bar{v}_{\omega\epsilon}, \dot{\bar{v}}_{q\epsilon}, \dot{\bar{v}}_{\omega\epsilon}) + \bar{\tau}_d/\underline{\lambda}_I}{k - (\hat{a}(\bar{v}_{\omega\epsilon}) + \hat{c}(\bar{v}_{q\epsilon}) + \hat{e}(\bar{v}_{q\epsilon}))},$$

by continuity of $\hat{a}$ to $\hat{f}$ in their arguments. Similar to Algorithm 1, define $\bar{q}_0^*(\epsilon) = 1$, and define the sequences

$$\begin{aligned} \bar{v}_{\tilde{\omega}}^{i*}(\epsilon) &= \bar{v}_{\omega\epsilon} + \bar{q}_{i-1}^*(\epsilon)\lambda\left[1 - \sqrt{1 - \bar{v}_{q\epsilon}^2}\right] \\ &\quad + [2\bar{w}_d + \lambda]\,\bar{v}_{q\epsilon}, \\ \bar{r}_i^*(\epsilon) &= \Omega_\wedge + \bar{v}_{\tilde{\omega}}^{i*}(\epsilon), \\ \bar{q}_i^*(\epsilon) &= \frac{\bar{r}_i^*(\epsilon)}{\lambda}. \end{aligned}$$

It is readily seen that $\bar{r}_i^*(\epsilon)$ and $\bar{q}_i^*(\epsilon)$ are all strictly increasing continuous functions for small enough $\epsilon > 0$, with

$$\lim_{\epsilon\to 0^+} \bar{r}_i^*(\epsilon) = \bar{r}_i, \quad \lim_{\epsilon\to 0^+} \bar{q}_i^*(\epsilon) = \bar{q}_i,$$

for all $i \ge 1$. Finally, in addition to the previous considerations on ϵ, let $\epsilon > 0$ be small enough such that $0 \le \bar{q}_1^*(\epsilon) < 1$. Proposition 1 may now be applied from time T onwards to give

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\le \bar{q}_i(\epsilon), \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\le 2\bar{r}_i(\epsilon), \end{aligned}$$

for all $i \ge 1$. Since $\epsilon > 0$ can be arbitrarily chosen, it must be that

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\le \bar{q}_i, \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\le 2\bar{r}_i, \end{aligned}$$

for all $i \ge 1$. Finally, since $\bar{q} = \lim_{i\to\infty} \bar{q}_i = \inf \bar{q}_i$ and $\bar{r} = \lim_{i\to\infty} \bar{r}_i = \inf \bar{r}_i$, it must be that

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\le \bar{q}, \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\le 2\bar{r}. \end{aligned}$$

This concludes the proof.

Remark 2 The proof of Theorem 1 shows that the errors may exhibit exponential growth before the estimation errors become small enough for the tracking errors to converge. Exponential growth in errors is clearly not desirable behavior. However, this is the worst case scenario, and may not occur in practice. Theorem 1 gives a result for the steady-state behavior of the closed-loop system and says nothing about the transient behavior. The result in Theorem 1 was obtained using only the observer properties given in Assumption 1, with no further assumption on the structure of the observer. As such, the result in Theorem 1 is very broadly applicable with regards to the convergence result and the ultimate bounds. However, for a given observer, improved understanding of the transient behavior of the closed-loop may be obtained by performing a coupled analysis of the observer together with the controller, such as is done in for example [12] and [13] (in the absence of measurement errors and disturbance torques). If there is any doubt as to the transient behavior, one can always wait a period of time before switching on the controller, until the observer has converged sufficiently.

Remark 3 The adaptive controllers in this paper estimate 24 parameters $\boldsymbol{\theta} \in \mathbb{R}^{18}$ and $\boldsymbol{\theta}_p \in \mathbb{R}^{6}$, while the unknown symmetric inertia matrix $\mathbf{I} \in \mathbb{R}^{3\times 3}$ has only 6 independent parameters. Thus, the state-space is significantly over-parameterized, which may lead to slow convergence modes in the closed-loop system. On the other hand, this over-parameterization allows one to obtain a dead-zone boundary (which is a level set of a Lyapunov-like function, see [17]), which is independent of the unknown inertia matrix, and can be computed based on state estimates alone as can be seen in (28) and (29). Thus, it can be practically implemented.

V. Concluding Remarks

This paper has considered observer-based adaptive attitude control for fully actuated rigid spacecraft with bounded disturbance torques. In particular, an adaptive attitude tracking control law has been considered, together with any observer that satisfies some very mild properties, including that the estimation errors are ultimately bounded. It has been shown that with such an observer, the attitude tracking errors resulting from the observer-based adaptive attitude tracking control law implementations are ultimately bounded. As such, a separation-type property has been obtained between the attitude control and observer designs. In particular, the steady-state properties of the closed-loop tracking errors depend only on the steady-state properties of the observer. Given known ultimate bounds on the estimation errors together with known bounds on the disturbance torques, a sequence of successively less conservative tracking error ultimate bounds has been obtained.

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