An LMI-Based Solution to Wahba's Problem

James R. Forbes[1]
University of Michigan, Ann Arbor, Michigan, 48109-2140 USA

Anton H. J. de Ruiter[2]
Ryerson University, Toronto, Ontario, M5B 2K3, Canada

I. Introduction

Wahba's problem was introduced in 1965 by Grace Wahba [1], and an important problem in aerospace engineering that typically involves finding an optimal rotation to fit a series of vector measurements. There have been many different methods developed to solve Wahba's problem, both directly in terms of the rotation matrix [2, 3], and in terms of the unit quaternion [4], the most famous method being QUEST [5]. A good survey of the different methods may be found in [6] and the references therein. Recently, a useful generalization of Wahba's problem has been made, allowing the determination of both attitude and body-rate using a time-history of vector measurements (see [7] to [9]). However, this note does not examine this problem, and treats only Wahba's original problem.

This note presents a new characterization of the solution to Wahba's problem, directly in terms of the rotation matrix. It is shown that under a mild condition (that is satisfied in many practical applications), Wahba's problem may be recast as a convex linear matrix inequality (LMI) optimization problem. This opens the door to a whole new class of solvers for Wahba's problem. This is accomplished by relaxing the non-convex special orthogonal group ($SO(3)$) constraint on the rotation matrix, to a convex LMI constraint. This constraint relaxation approach has applications beyond the solution of Wahba's problem, and can potentially be useful for other optimization problems involving vehicle attitude, such as guidance and control problems.

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[1] Assistant Professor, Department of Aerospace Engineering, FXB Building, 1320 Beal Street, Member AIAA.
[2] Assistant Professor, Department of Aerospace Engineering, 350 Victoria Street, Senior Member AIAA

The remainder of the note is organized as follows. Section 2 presents an overview of Wahba's problem and its solution in terms of the singular value decomposition [3]. Section 3 demonstrates that under a mild condition, the Wahba problem may be recast as an LMI problem, leading to an identical solution, and conditions are investigated under which this mild condition is satisfied. Section 4 presents a pair of numerical examples comparing the LMI-based solution to existing well-established solutions to Wahba's problem. Section 5 contains concluding remarks. The appendix contains a technical mathematical result, which is used in the note.

II. Wahba's Problem and Solution

Wahba's problem was originally posed by Grace Wahba in 1965 [1]. This section presents a brief overview of Wahba's problem, a well-known reformulation, and its solution based upon the singular value decomposition. The singular value decomposition solution is reviewed because it will be referenced when the LMI-based solution is derived in the next section.

Problem 1 (Wahba's Problem)**.** *Given N vectors, $\mathbf{s}_{b,k}^{m} \in \mathbb{R}^3$, with corresponding vectors $\mathbf{s}_{I,k} \in \mathbb{R}^3$, Wahba's Problem is to find the matrix $\mathbf{C} \in SO(3)$, where*

$$SO(3) = \left\{ \mathbf{C} \in \mathbb{R}^{3\times 3} : \mathbf{C}^T\mathbf{C} = \mathbf{1}, \det \mathbf{C} = +1 \right\},$$

to minimize the cost function

$$J = \sum_{k=1}^{N} w_k \left(\mathbf{s}_{b,k}^{m} - \mathbf{C}\mathbf{s}_{I,k} \right)^T \left(\mathbf{s}_{b,k}^{m} - \mathbf{C}\mathbf{s}_{I,k} \right), \tag{1}$$

where $0 < w_k < \infty$ are positive weights for $k = 1, \ldots, N$.

Problem 1 is readily shown to be equivalent to solving the minimization problem [6]:

$$\text{minimize } \hat{J} = -\text{tr}\left[\mathbf{C}\mathbf{B}^T\right] \text{ subject to } \mathbf{C} \in SO(3), \tag{2}$$

where

$$\mathbf{B}^T = \sum_{k=1}^{N} w_k \mathbf{s}_{I,k} \mathbf{s}_{b,k}^{mT}. \tag{3}$$

In some instances, it may be desirable to orthonormalize a given matrix $\mathbf{D} \in \mathbb{R}^{3\times 3}$. For example, when $\mathbf{C} \in SO(3)$, its kinematics satisfy Poisson's equation $\dot{\mathbf{C}} = -\boldsymbol{\omega}^{\times}\mathbf{C}$ [11, ch. 2], with initial condition $\mathbf{C}(t_0) \in SO(3)$. Direct numerical integration will result in a solution $\hat{\mathbf{C}}(t)$ that is no longer in $SO(3)$ due to numerical inaccuracies. It is therefore desirable to orthonormalize $\hat{\mathbf{C}}(t)$ after each numerical integration step. Orthonormalization of a rotation matrix estimate $\hat{\mathbf{C}}(t)$ can be thought of in the same manner as normalization of a quaternion estimate.

Considering the matrix $\mathbf{D}$ to be an approximation of a matrix $\mathbf{C} \in SO(3)$, it is reasonable to expect $\det[\mathbf{D}] > 0$ (otherwise it would be a very poor approximation). It would be desirable to orthonormalize the matrix $\mathbf{D} \in \mathbb{R}^{3\times 3}$ in an optimal manner, as stated in the next problem. Note that this is very similar to the orthogonal Procrustes problem [10], in which $\mathbf{C}$ is only required to be orthonormal, without any restriction on the sign of its determinant.

Problem 2 (Matrix Orthonormalization). *Let* $\mathbf{D} \in \mathbb{R}^{3\times 3}$*, with* $\det \mathbf{D} > 0$*,*

$$\text{minimize } J = tr\left[(\mathbf{D} - \mathbf{C})^T (\mathbf{D} - \mathbf{C})\right], \tag{4}$$

subject to $\mathbf{C} \in SO(3)$*.*

In fact, since $\det \mathbf{D} > 0$, the solution of Problem 2 is identical to the orthogonal Procrustes problem.

Analogously to Wahba's problem (Problem 1), it is readily shown that the minimization problem in (4) is equivalent to the minimization problem

$$\text{minimize } \hat{J} = -\text{tr}\left[\mathbf{C}\mathbf{D}^T\right] \text{ subject to } \mathbf{C} \in SO(n). \tag{5}$$

Comparing (2) and (5), it can be seen that Problems 1, 2 are identical in form. As such, only Problem 1 shall be considered from this point on.

It is well known that the set of all solutions to Problem 1 is given by [3]

$$\mathbf{C} = \mathbf{V}\text{diag}\left\{1, 1, \det \mathbf{V} \det \mathbf{U}\right\} \mathbf{U}^T, \tag{6}$$

where $\mathbf{V}$ and $\mathbf{U}$ are obtained from a singular value decomposition of $\mathbf{B}$ such that

$$\mathbf{B} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^T, \tag{7}$$

where $\mathbf{V}^T\mathbf{V} = \mathbf{1}$, $\mathbf{U}^T\mathbf{U} = \mathbf{1}$ and $\boldsymbol{\Sigma} = \text{diag}\left\{\sigma_1, \sigma_2, \sigma_3\right\}$ with $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. Furthermore, the solution is unique when $\det \mathbf{B} > 0$, or $\text{rank}[\mathbf{B}] = 2$.

From (6), it can be seen that

$$\mathbf{C} = \mathbf{V}\mathbf{U}^T, \text{ if } \det \mathbf{B} > 0, \tag{8}$$

and

$$\mathbf{C} = \mathbf{V}\text{diag}\left\{1, 1, -1\right\}\mathbf{U}^T, \text{ if } \det \mathbf{B} < 0. \tag{9}$$

III. An LMI-based Solution

Relax the constraint in (2) (which is equivalent to Problem 1) to obtain the new problem:

Problem 3. *Let* $\mathbf{B} \in \mathbb{R}^{n\times n}$*.*

$$\textit{minimize } \hat{J} = -tr\left[\mathbf{C}\mathbf{B}^T\right] \textit{ subject to } \|\mathbf{C}\| \leq 1, \tag{10}$$

where $\|\mathbf{A}\| = \sigma_{max}(\mathbf{A})$ *is the induced* $2-$*norm of the matrix* $\mathbf{A}$*, and* σ_{max} *the maximum singular value.*

Note that $\|\mathbf{C}\| = 1$ for all $\mathbf{C} \in SO(3)$, so the constraint set in Problem 3 includes the constraint set in (2).

The constraint set in Problem 3 is compact. As such, a global minimizing solution exists. Furthermore, since the cost function $\hat{J}$ is linear in $\mathbf{C}$, the minimizing solution must lie on the boundary of the constraint set. That is, the minimizing solution must satisfy $\|\mathbf{C}\| = 1$.

Next, consider any $\mathbf{C}_1$ and $\mathbf{C}_2$ satisfying $\|\mathbf{C}_1\| \leq 1$ and $\|\mathbf{C}_2\| \leq 1$, and form the convex combination $\alpha\mathbf{C}_1 + (1-\alpha)\mathbf{C}_2$ for some $\alpha \in [0, 1]$. Then,

$$\|\alpha\mathbf{C}_1 + (1-\alpha)\mathbf{C}_2\| \leq \alpha\|\mathbf{C}_1\| + (1-\alpha)\|\mathbf{C}_2\| \leq 1,$$

which shows that the constraint set is convex. Consequently, any local minimizing solution of (10) must be a global minimizing solution. The minimizing solutions are now found in the cases rank$[\mathbf{B}] = 3$, and rank$[\mathbf{B}] = 2$.

Consider a singular value decomposition of $\mathbf{B}$, as given in (7). Since $\mathbf{V}$ and $\mathbf{U}$ are nonsingular, without loss of generality one may write

$$\mathbf{C} = \mathbf{V}\mathbf{S}\mathbf{U}^T, \tag{11}$$

for some $\mathbf{S} \in \mathbb{R}^{3\times 3}$ (e.g., simply set $\mathbf{S} = \mathbf{V}^T\mathbf{C}\mathbf{U}$ given $\mathbf{C}$). Then, Problem 3 is equivalent to

$$\text{minimize } \hat{J} = -\text{tr}\left[\mathbf{S}\mathbf{\Sigma}\right] \text{ subject to } \|\mathbf{S}\| \leq 1. \tag{12}$$

Denote the ij^{th} term of $\mathbf{S}$ by s_{ij}. Then, the cost function in (12) becomes

$$\hat{J} = -\sum_{i=1}^{3} s_{ii}\sigma_i. \tag{13}$$

By Corollary 1 (see Appendix A), if $\|\mathbf{S}\| \leq 1$ it must be that $|s_{ii}| \leq 1$. As such, one has

$$\hat{J}(\mathbf{S}) \geq -\sum_{i=1}^{3} \sigma_i, \;\; \forall \mathbf{S} \text{ satisfying } \|\mathbf{S}\| \leq 1. \tag{14}$$

Noting that $\mathbf{S} = \mathbf{1}$ is a member of the constraint set, the lower bound in (14) is in fact the global minimum for (12). Therefore, all minimizing $\mathbf{S}$ of (12) must have

$$s_{ii} = 1 \text{ if } \sigma_i > 0.$$

Correspondingly, by Proposition 1 (see Appendix A) they must have

$$s_{ij} = s_{ji} = 0 \text{ for } j \neq i \text{ if } \sigma_i > 0.$$

Hence, if rank$[\mathbf{B}] \geq 2$, any minimizing $\mathbf{S}$ of (12) takes the form

$$\mathbf{S} = \text{diag}\{1, 1, s_{33}\}, \tag{15}$$

where

$$s_{33} = 1, \quad \text{if rank}[\mathbf{B}] = 3,$$

and

$$|s_{33}| \leq 1, \quad \text{if rank}[\mathbf{B}] = 2.$$

When rank$[\mathbf{B}] = 2$ (and thus $\sigma_{33} = 0$) then s_{33} has no effect on the cost function given in (13), and as such s_{33} may arbitrarily be chosen subject to the norm-constraint $\|\mathbf{S}\| \leq 1$.

Case 1: rank$[\mathbf{B}] = 3$

When $\mathbf{B}$ has full rank, from (16) and (11), the minimizing $\mathbf{C}$ for Problem 3 is unique, and is given by

$$\mathbf{C} = \mathbf{V}\mathbf{U}^T. \tag{16}$$

Comparing this to (8), it can be seen that it coincides with the solution to Wahba's problem (Problem 1) when $\det \mathbf{B} > 0$. However, it does not coincide when $\det \mathbf{B} < 0$, (compare (16) to (9)).

Case 2: rank$[\mathbf{B}] = 2$

When rank$[\mathbf{B}] = 2$, from (16) and (11), the minimizing solutions of Problem 3 are non-unique, and are given by

$$\mathbf{C} = \mathbf{V}\text{diag}\{1, 1, s_{33}\}\mathbf{U}^T, \tag{17}$$

where

$$|s_{33}| \leq 1, \quad \text{if rank}[\mathbf{B}] = 2.$$

Consequently, Problem 3 is only equivalent to Wahba's problem (Problem 1) when $\det \mathbf{B} > 0$.

Problem 3 shall now be reformulated as a convex optimization problem with a LMI constraint. To this end, note that the norm constraint in Problem 3 is equivalent to

$$\mathbf{C}^T\mathbf{C} \leq \mathbf{1}. \tag{18}$$

Recall the Schur complement [13, ch. 2]: for $\mathbf{\Phi}_{11} = \mathbf{\Phi}_{11}^T \in \mathbb{R}^{p\times p}$, $\mathbf{\Phi}_{12} \in \mathbb{R}^{p\times q}$, $\mathbf{\Phi}_{21} \in \mathbb{R}^{q\times p}$, $\mathbf{\Phi}_{22} = \mathbf{\Phi}_{22}^T \in \mathbb{R}^{q\times q}$ where $\mathbf{\Phi}_{22} > \mathbf{0}$, then

$$\mathbf{\Phi}_{11} - \mathbf{\Phi}_{12}\mathbf{\Phi}_{22}^{-1}\mathbf{\Phi}_{21} \geq \mathbf{0} \quad \Leftrightarrow \quad \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{21} & \mathbf{\Phi}_{22} \end{bmatrix} \geq \mathbf{0}.$$

Therefore, setting $\mathbf{\Phi}_{11} = \mathbf{1}$, $\mathbf{\Phi}_{12} = \mathbf{C}^T$, $\mathbf{\Phi}_{21} = \mathbf{C}$, $\mathbf{\Phi}_{22} = \mathbf{1}$ and using the Schur complement, (18) is in turn equivalent to the LMI

$$\begin{bmatrix} \mathbf{1} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{1} \end{bmatrix} \geq \mathbf{0}.$$

Therefore, Problem 3 may be recast as the following LMI problem.

Problem 4. *Let* $\mathbf{B} \in \mathbb{R}^{n\times n}$. *Find* $\mathbf{C} \in \mathbb{R}^{n\times n}$ *to*

$$\textit{minimize } \hat{J} = -tr\left[\mathbf{C}\mathbf{B}^T\right], \tag{19}$$

subject to

$$\begin{bmatrix} \mathbf{1} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{1} \end{bmatrix} \geq \mathbf{0}. \tag{20}$$

The previous analysis is now summarized in the following Theorem.

Theorem 1. *Problem 4 with* $\det \mathbf{B} > 0$ *has a unique global minimum, with no other local minima. In this case, the solution of Problem 4 is equal to the unique solution of Problem 1.*

Problem 4 may be easily solved using existing LMI solvers.

The limitation of requiring $\det[\mathbf{B}] > 0$ for the LMI-based solution to Wahba's problem is now examined. Suppose that the measured vectors $\mathbf{s}_{b,k}^m \in \mathbb{R}^n$ are generated according to

$$\mathbf{s}_{b,k}^m = \mathbf{s}_{b,k} + \mathbf{v}_k, \tag{21}$$

where

$$\mathbf{s}_{b,k} = \mathbf{C}\mathbf{s}_{I,k}, \tag{22}$$

and $\mathbf{v}_k \in \mathbb{R}^3$ is a measurement error. Then, (3) becomes

$$\mathbf{B}^T = \bar{\mathbf{B}}^T + \Delta\mathbf{B}^T, \tag{23}$$

where

$$\bar{\mathbf{B}}^T = \sum_{k=1}^{N} w_k \mathbf{s}_{I,k}\mathbf{s}_{b,k}^T, \;\; \Delta\mathbf{B}^T = \sum_{k=1}^{N} w_k \mathbf{s}_{I,k}\mathbf{v}_k^T. \tag{24}$$

Rewrite $\bar{\mathbf{B}}^T$ and $\Delta\mathbf{B}^T$ in (24) as

$$\bar{\mathbf{B}}^T = \mathbf{S}_I \bar{\mathbf{W}} \mathbf{S}_b^T, \;\; \Delta\mathbf{B}^T = \mathbf{S}_I \bar{\mathbf{W}} \bar{\mathbf{V}}^T \tag{25}$$

where

$$\mathbf{S}_I = \begin{bmatrix} \mathbf{s}_{I,1} & \cdots & \mathbf{s}_{I,N} \end{bmatrix}, \;\; \bar{\mathbf{W}} = \mathrm{diag}\left\{w_1, \ldots, w_N\right\}, \;\; \mathbf{S}_b = \begin{bmatrix} \mathbf{s}_{b,1} & \cdots & \mathbf{s}_{b,N} \end{bmatrix},$$

and

$$\bar{\mathbf{V}} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_N \end{bmatrix}.$$

From (22), one obtains

$$\mathbf{S}_b = \mathbf{C}\mathbf{S}_I.$$

Therefore,

$$\bar{\mathbf{B}}^T = \mathbf{S}_I \mathbf{W} \mathbf{S}_I^T \mathbf{C}^T. \tag{26}$$

Consequently, one has

$$\det \bar{\mathbf{B}} = \det(\mathbf{S}_I \bar{\mathbf{W}} \mathbf{S}_I^T) \det \mathbf{C}.$$

When $\bar{\mathbf{B}}$ has full rank, the matrix $\mathbf{S}_I \bar{\mathbf{W}} \mathbf{S}_I^T$ is positive definite, and

$$\operatorname{sign}[\det \bar{\mathbf{B}}] = \operatorname{sign}[\det \mathbf{C}]. \tag{27}$$

Finally, by continuity of the determinant and (23) together with (25), it is concluded that for a given $\mathbf{S}_I$ and weight $\bar{\mathbf{W}}$, there exists $\delta > 0$ such that

$$\operatorname{sign}[\det \mathbf{B}] = \operatorname{sign}[\det \mathbf{C}], \ \ \forall \bar{\mathbf{V}} \in \mathbb{R}^{n \times N} \text{ such that } \|\bar{\mathbf{V}}\| < \delta. \tag{28}$$

That is, if the collection of vectors $\mathbf{s}_{i,k}$ is geometrically rich enough, and the measurement errors $\mathbf{v}_k$ are small enough, $\det \mathbf{B}$ will have the same sign as $\det \mathbf{C}$. Clearly, there must therefore be at least three vector measurements. On the other-hand, if the objective is to solve Problem 2 then the determinant condition is automatically satisfied.

IV. Numerical Examples

The LMI-based solution to Wahba's problem is now demonstrated with a pair of numerical examples. In the first example, a noise-free set of measurement vectors is used, demonstrating that the LMI-based method returns the original rotation matrix, which in this case is the known optimal solution to Wahba's problem. In the second example, a set of noise-corrupted measurement vectors is used. In this case, the LMI-based solution is compared to the SVD-based solution in (6), as well as other well-established solution methods including the q-method [12, ch. 12], QUEST [5], and ESOQ2 [17], each of which returns the same solution. All numerical work is done on a MacBook Pro with a 2.3 GHz Intel Core i5 processor and 4 GB of RAM running MATLAB 7.12.0 (R2011a).

Both examples use the following vectors and weights:

$$\mathbf{s}_{I,1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{s}_{I,2} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{s}_{I,3} = \frac{1}{\sqrt{26}} \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{s}_{I,4} = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{s}_{I,5} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and

$$w_k = \frac{1}{\sigma_i^2}, \quad k = 1 \ldots 5,$$

where

$$\sigma_1 = 0.0100, \quad \sigma_2 = 0.0325, \quad \sigma_3 = 0.0550, \quad \sigma_4 = 0.0775, \quad \sigma_5 = 0.1000.$$

Assume the true attitude is given by

$$\mathbf{C} = \mathbf{C}_3(60^\circ)\mathbf{C}_2(-30^\circ)\mathbf{C}_1(45^\circ) = \begin{bmatrix} 0.4330 & 0.4356 & 0.7891 \\ -0.7500 & 0.6597 & 0.0474 \\ -0.5000 & -0.6124 & 0.6124 \end{bmatrix},$$

where $\mathbf{C}_1(\cdot)$, $\mathbf{C}_2(\cdot)$, and $\mathbf{C}_3(\cdot)$ are principal rotations about the 1, 2, and 3 axes, respectively [11, ch. 2].

Case 1 - Noise Free Measurements

Consider the case where the vector measurements are not corrupted by any noise, that is $\mathbf{s}_{b,k}^m = \mathbf{C}\mathbf{s}_{I,k}$. In this case, it is simple to show that rank$[\mathbf{B}] = 3$ and det$[\mathbf{B}] > 0$, and as such, the solution to Problem 4 equals the unique solution to Problem 1. Using the software `YALMIP` [14] and `SeDuMi` [15] to solve Problem 4, $\mathbf{C}$ is exactly recovered, as expected.

Case 2 - Noisy Measurements

Next consider the case where the vector measurements are corrupted by noise. Specifically, consider the following vector measurements:

$$\mathbf{s}_{b,1}^m = \begin{bmatrix} 0.9082 \\ 0.3185 \\ 0.2715 \end{bmatrix}, \quad \mathbf{s}_{b,2}^m = \begin{bmatrix} 0.5670 \\ 0.3732 \\ -0.7343 \end{bmatrix}, \quad \mathbf{s}_{b,3}^m = \begin{bmatrix} -0.2821 \\ 0.7163 \\ 0.6382 \end{bmatrix},$$

$$\mathbf{s}_{b,4}^m = \begin{bmatrix} 0.7510 \\ -0.3303 \\ 0.5718 \end{bmatrix}, \quad \mathbf{s}_{b,5}^m = \begin{bmatrix} 0.9261 \\ -0.2053 \\ -0.3166 \end{bmatrix},$$

Again, it is straight forward to verify that $\text{rank}[\mathbf{B}] = 3$ and $\det[\mathbf{B}] > 0$. As such, the solution to Problem 4 equals the unique solution to Problem 1. Using the software `YALMIP` and `SeDuMi` to solve Problem 4, the best estimate of $\mathbf{C}$, called $\hat{\mathbf{C}}$, is found to be

$$\hat{\mathbf{C}} = \begin{bmatrix} 0.4153 & 0.4472 & 0.7921 \\ -0.7562 & 0.6537 & 0.0274 \\ -0.5056 & -0.6104 & 0.6097 \end{bmatrix}.$$

Define the error between $\mathbf{C}$ and $\hat{\mathbf{C}}$ to be $\mathbf{C}_e = \hat{\mathbf{C}}\mathbf{C}^T$. Recall that any element of $SO(3)$ can be expressed in terms of an Euler axis and and Euler angle [11, ch. 2]. In order to assess how close $\hat{\mathbf{C}}$ is to $\mathbf{C}$, the Euler angle associated with $\mathbf{C}_e$ will be computed. The Euler angle of $\mathbf{C}_e$ is $\phi_e = 1.27^\circ$ where $\cos\phi_e = \cos(0.5(\text{trace}\mathbf{C}_e - 1))$, indicating that $\hat{\mathbf{C}}$ is a good estimate of $\mathbf{C}$.

Using either the q-method [12, ch. 12], QUEST [5], the SVD method [3], or ESOQ2 [17] to find $\hat{\mathbf{C}}$ yields the same result as the LMI solution presented. This is expected, as they are each a solution to the same problem, that being Problem 1. In Table 1 is the execution time of q-method, QUEST, the SVD method, and ESOQ2. The execution time is computed using `MATLAB`'s "`tic`" and "`toc`" functions. As expected, ESOQ2 is the fastest algorithm [6], followed by QUEST, the q-method, the SVD method, and finally the proposed LMI method.

The fact that the LMI method is the slowest is due to the fact that the `YALMIP` and `SeDuMi` have been used; these are general tools used to solve LMI problems. ESOQ2, the q-method, and QUEST, are taylored to find the quaternion representing the attitude, and as such, their execution

Algorithm	q-method	QUEST	SVD Method	ESOQ2	LMI
Time (s)	0.00143210	0.00077290	0.00307701	0.000607732	0.06622991

Table 1 Execution time of q-method, QUEST, the SVD method, and ESOQ2. Execution time computed using `MATLAB`'s "`tic`" and "`toc`" functions.

time is much faster. Additionally, the q-method uses `MATLAB`'s custom eigenvalue solver and the SVD method uses `MATLAB`'s custom singular value decomposition solver, both of which are highly optimized. Future work will focus on designing a custom computer code to solve Problem 4, exploiting the specific structure of the objective function and LMI constraint. A custom computer code is expected to decrease the execution time significantly. As discussed in [18, 19], a custom computer code can be executed 5 to 600 times faster than a standard code, depending on the application.

Remark

Although the LMI form of the constraint given in (20) has been used to solve for the attitude in Wahba's problem, (20) can be used as a constraint in other optimization problems, such as guidance and control problems involving attitude. For example, in the domain of Mars powered-decent guidance, convex optimization methods are often employed [20]. Future work will investigate the use of (20) in guidance and control problems.

V. Conclusion

This note has presented a new characterization of the solution to the famous Wahba problem. It has been shown that when a mild condition is satisfied (which is demonstrated to hold for many practical problems of interest), the Wahba problem can be recast as a linear matrix inequality (LMI) optimization problem. This opens the door to a whole new class of solution methods for these types of Wahba problems. Equivalence between the Wahba problem and the LMI problem is accomplished by relaxing the non-convex constraint on the rotation matrix $\mathbf{C} \in SO(3)$, creating instead a convex constraint of the form $\|\mathbf{C}\| \leq 1$, which is equivalently represented in LMI form. While this approach has been demonstrated for the Wahba problem, it has applicability to other optimization problems involving attitude, such as guidance and navigation problems.

Appendix A

Proposition 1. *Consider any matrix* $\mathbf{A} \in \mathbb{R}^{n \times m}$*, with* $\|\mathbf{A}\| = \ell$*, for some* $\ell \geq 0$*. Denote the* ij^{th} *term of* $\mathbf{A}$ *as* a_{ij}*. Then,*

$$\sqrt{\sum_{i=1}^{n} a_{ij}^2} \leq \ell, \;\; j = 1, \ldots, m. \tag{29}$$

and

$$\sqrt{\sum_{j=1}^{m} a_{ij}^2} \leq \ell, \;\; i = 1, \ldots, n, \tag{30}$$

Proof. By definition,

$$\|\mathbf{A}\| = \max_{\|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2,$$

where $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T \mathbf{x}}$ denotes the vector 2-norm. Let us now set $\mathbf{x} = \mathbf{e}_j = [e_{j,1}, \ldots, e_{j,m}]^T$, for some $j \in \{1, \ldots, m\}$, where

$$e_{j,k} = \begin{cases} 1, & k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\|\mathbf{x}\|_2 = 1$, and

$$\mathbf{A}\mathbf{x} = [a_{1,j}, \ldots, a_{n,j}]^T .$$

Then, by definition of the matrix norm above, one must have

$$\|\mathbf{A}\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} a_{ij}^2} \leq \|\mathbf{A}\| = \ell,$$

which is (29). Since $\|\mathbf{A}\| = \|\mathbf{A}^T\|$, repeating the above argument for $\mathbf{A}^T$ yields (30). □

The following Corollary is an immediate consequence of Proposition 1.

Corollary 1. *Consider any matrix* $\mathbf{A} \in \mathbb{R}^{n \times m}$*, with* $\|\mathbf{A}\| = \ell$*, for some* $\ell \geq 0$*. Denote the* ij^{th} *term of* $\mathbf{A}$ *as* a_{ij}*. Then,*

$$|a_{ij}| \leq \ell.$$

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