Spacecraft Attitude Tracking with Guaranteed Performance Bounds

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1 Introduction

Model uncertainties and measurement errors are very important factors that practical spacecraft attitude control systems are subject to. Adaptive control is a well known method for dealing with modeling uncertainty [1, 2, 3]. Recently new control laws have been obtained that treat disturbances and model uncertainties [4, 5, 6, 7, 8]. References [4, 5] deal with the attitude regulation problem only. References [6, 7] both present globally convergent control laws for the attitude tracking problem, when bounds on the spacecraft inertia matrix and the disturbances are known. The advantage of these approaches is that the form of the disturbance need not be known, only the bound. In [8], the inertia matrix and linearly parameterizable disturbances are estimated adaptively. On the other hand, all of the afore-mentioned works are based on the availability of perfect measurements. Reference [9] explicitly considers measurement errors and studies performance, given bounds on the measurement error in the context of a model reference adaptive controller. This is a very important issue that any practical control system design must address. In particular, guaranteed performance bounds will be very useful for the control system design if performance specifications are given. There are well-established techniques to obtain these bounds for linear systems, however they are generally lacking for the more general nonlinear case (reference [9] being an exception). In practise, extensive simulation-based Monte-Carlo analyses are used to determine closed-loop performance, which can be quite time-consuming, particularly if they are used to determine suitable control gains.

In this note, we consider non-adaptive and adaptive attitude tracking. Guaranteed analytical performance bounds are obtained in the presence of model uncertainties and measurement errors. The bounds can be useful for attitude control system designers to assist in gain selection given steady-state performance specifications, thus reducing the

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need for time-consuming Monte-Carlo analyses.

The note is organised as follows. First, a result on the filtered error from [6] is generalized. It is shown that if the filtered error is ultimately upper bounded with known bound, then the attitude and body-rate errors are also ultimately upper bounded. Subsequently, making use of this result together with sequential Lyapunov-type analyses, bounds on the steady-state tracking errors are derived when bounded model uncertainties and measurement errors are present.

2 Mathematical Preliminaries

In this note, the vector and matrix norms used are $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ and $\|\mathbf{X}\| = \sqrt{\lambda_{max}(\mathbf{X}^T\mathbf{X})}$ (where $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue), respectively. The identity matrix will be denoted by $\mathbf{1}$. We will denote the unit quaternion by $(\mathbf{q}, q_4)$, where $\mathbf{q} \in \mathbb{R}^3$ is the vector part of the quaternion, and $q_4 \in \mathbb{R}$ is the scalar part. Associated with a vector $\mathbf{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T \in \mathbb{R}^3$ is the matrix

$$\mathbf{a}^\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

The following result on the filtered error is required in this note.

Lemma 1

Consider $\mathbf{r}(t) \triangleq \boldsymbol{\omega}(t) \pm \boldsymbol{\Lambda}\mathbf{q}(t)$, *where* $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T > \mathbf{0}$ *is positive definite,* $\boldsymbol{\omega}(t) : \mathbb{R}^+ \to \mathbb{R}^3$ *is continuous, and* $\mathbf{q}(t) : \mathbb{R}^+ \to \mathbb{R}^3$ *together with* $q_4(t) : \mathbb{R}^+ \to \mathbb{R}$ *make up a unit quaternion satisfing the kinematical equations*

$$\begin{aligned} \dot{\mathbf{q}} &= \tfrac{1}{2}(\mathbf{q}^\times + q_4\mathbf{1})\boldsymbol{\omega}, \\ \dot{q}_4 &= -\tfrac{1}{2}\mathbf{q}^T\boldsymbol{\omega}, \end{aligned} \tag{1}$$

with initial conditions $\mathbf{q}(0)^T\mathbf{q}(0) + q_4^2(0) = 1$.

If there exist an $\bar{r} \geq 0$ *and a finite* $T \geq 0$ *such that* $\|\mathbf{r}(t)\| \leq \bar{r}$ *for all* $t \geq T$, *then*

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\leq \frac{\bar{r}}{\lambda_{min}(\boldsymbol{\Lambda})}, \\ \limsup_{t\to\infty} \|\boldsymbol{\omega}(t)\| &\leq \left(\frac{\lambda_{max}(\boldsymbol{\Lambda})}{\lambda_{min}(\boldsymbol{\Lambda})} + 1\right)\bar{r}. \end{aligned}$$

Proof

Using the function $V = \mathbf{q}^T\mathbf{q} + q_4^2$ together with the kinematics (1), it is readily shown that $\mathbf{q}(t)^T\mathbf{q}(t) + q_4^2(t) = 1$ for all $t \in \mathbb{R}$. Consequently, $\mathbf{q}^T(t)\mathbf{q}(t) \leq 1$ and $q_4^2(t) \leq 1$ for all $t \in \mathbb{R}$. If $\bar{r}/\lambda_{min}(\boldsymbol{\Lambda}) \geq 1$, then the result is trivially obtained. Therefore, we shall assume that $\bar{r}/\lambda_{min}(\mathbf{\Lambda}) < 1$.

We shall treat the case $\mathbf{r} = \boldsymbol{\omega} + \mathbf{\Lambda q}$. Rearranging, we have $\boldsymbol{\omega} = -\mathbf{\Lambda q} + \mathbf{r}$. Substituting this into the second equation in (1), we obtain

$$\dot{q}_4 = \frac{1}{2}\mathbf{q}^T\mathbf{\Lambda q} - \frac{1}{2}\mathbf{q}^T\mathbf{r}.$$

Taking norms on the right-hand side, and using the assertion that $\|\mathbf{r}(t)\| \le \bar{r}$ for all $t \ge T$, we obtain the upper bound

$$-\dot{q}_4 \le -\frac{1}{2}\|\mathbf{q}\|\left[\lambda_{min}(\mathbf{\Lambda})\|\mathbf{q}\| - \bar{r}\right], \ \ \forall t \ge T.$$

Since $\mathbf{q}^T\mathbf{q} + q_4^2 = 1$, this can be rewritten as

$$\dot{q}_4 \ge \frac{1}{2}\left(1-q_4^2\right)^{\frac{1}{2}}\left[\lambda_{min}(\mathbf{\Lambda})\left(1-q_4^2\right)^{\frac{1}{2}} - \bar{r}\right], \ \ \forall t \ge T. \tag{2}$$

Choose an arbitrary $\epsilon > 0$ such that $\bar{r}/\lambda_{min}(\mathbf{\Lambda}) + \epsilon < 1$. Then, (2) shows that for all $t \ge T$,

$$\dot{q}_4 > \frac{1}{2}\left(\bar{r}/\lambda_{min}(\mathbf{\Lambda}) + \epsilon\right)\lambda_{min}(\mathbf{\Lambda})\epsilon > 0,$$

on the open interval $F_\epsilon \triangleq \left(-\left[1-\left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)^2\right]^{1/2}, \left[1-\left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)^2\right]^{1/2}\right)$. Outside the interval F_ϵ, $\dot{q}_4$ is sign indefinite. There are now two cases to consider: 1) $q_4(t) \in G_\epsilon \triangleq \left[-1, -\left(1-\left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)^2\right)^{1/2}\right] \cup \left[\left(1-\left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)^2\right)^{1/2}, 1\right]$ for all $t \ge T$, and 2) there exists a $t_1 \ge T$ such that $q_4(t_1) \in F_\epsilon$.

In case 1), $\mathbf{q}(t)^T\mathbf{q}(t) = 1 - q_4^2(t) \le \left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)^2$ for all $t \ge T$. In case 2), since $q_4(t_1) \in F_\epsilon$ and $\dot{q}_4 \ge \frac{1}{2}\left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)\lambda_{min}(\mathbf{\Lambda})\epsilon > 0$ on F_ϵ, there must exist a $T' > t_1$ such that $q_4(t)$ enters the interval $G_\epsilon^u \triangleq \left[\left(1-\left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)^2\right)^{1/2}, 1\right]$ at $t = T'$. Once inside the interval G_ϵ^u, $q_4(t)$ can never leave, since $\dot{q}_4 > 0$ on the lower boundary. Therefore, similar to case 1), $\mathbf{q}^T(t)\mathbf{q}(t) = 1 - q_4^2(t) \le \left(\bar{r}/\lambda_{min}(\mathbf{\Lambda})+\epsilon\right)^2$ for all $t \ge T'$. Since $\epsilon > 0$ is arbitrary, the conclusion is established for $\mathbf{q}(t)$. The conclusion for $\boldsymbol{\omega}(t)$ follows by noting that $\|\boldsymbol{\omega}\| \le \lambda_{max}(\mathbf{\Lambda})\|\mathbf{q}\| + \|\mathbf{r}\|$. Therefore, we have shown the result for $\mathbf{r} = \boldsymbol{\omega} + \mathbf{\Lambda q}$. The proof for $\mathbf{r} = \boldsymbol{\omega} - \mathbf{\Lambda q}$ is identical, with the inequality in (2) reversed. $\square$

We now show that Lemma 1 in [6] can be obtained from the above as a special case in the following corollary.

Corollary 1

Let the conditions of Lemma 1 hold. If in addition $\mathbf{r}(t) \to \mathbf{0}$ as $t \to \infty$, then $\mathbf{q}(t) \to \mathbf{0}$ and $\boldsymbol{\omega}(t) \to \mathbf{0}$ as $t \to \infty$ also.

Proof

Choose an arbitrary $\epsilon > 0$. Then, since $\mathbf{r}(t) \to \mathbf{0}$, there exists a finite $t_1 \geq 0$ such that for all $t \geq t_1$, $||\mathbf{r}(t)|| < \lambda_{min}(\mathbf{\Lambda})\epsilon/2$. Setting $\bar{r} = \lambda_{min}(\mathbf{\Lambda})\epsilon/2$, it follows by Lemma 1, there exists a finite $t_2 \geq t_1$ such that $||\mathbf{q}(t)|| \leq \bar{r}/\lambda_{min}(\mathbf{\Lambda}) + \epsilon/2 = \epsilon$ for all $t \geq t_2$. Therefore, since ϵ was arbitrary, the conclusion for $\mathbf{q}(t)$ follows. Finally, since $\boldsymbol{\omega}(t) = \mp\mathbf{\Lambda}\mathbf{q}(t) + \mathbf{r}(t)$, it must be that $\boldsymbol{\omega}(t) \to \mathbf{0}$ also. □

Remark 1

Given that both unit quaternions $(\mathbf{q}, q_4)$ and $(-\mathbf{q}, -q_4)$ represent the same attitude, it is not surprising that Lemma 1 holds for $\mathbf{r}(t) \stackrel{\Delta}{=} \boldsymbol{\omega}(t) \pm \mathbf{\Lambda}\mathbf{q}(t)$. From a practical attitude control standpoint, this means that the attitude control system is free to feed back either $\mathbf{q}$ or $-\mathbf{q}$, provided there is no switch from one to the other. Due to the unit norm constraint $\mathbf{q}^T\mathbf{q} + q_4^2 = 1$, any switch from $(\mathbf{q}, q_4)$ to $(-\mathbf{q}, -q_4)$ would result in a noticable discontinuity, which the feedback control logic could easily detect and correct.

3 Spacecraft Attitude Tracking Problem Formulation

The spacecraft attitude dynamics are given in body coordinates by [10, p. 59]

$$\mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\tau}_c + \boldsymbol{\tau}_d, \tag{3}$$

where $\mathbf{I}$ is the spacecraft inertia matrix, $\boldsymbol{\omega}$ is the angular velocity relative to an inertial frame, $\boldsymbol{\tau}_c$ is the control torque and $\boldsymbol{\tau}_d$ is an external disturbance torque.

The spacecraft inertia matrix $\mathbf{I}$ is assumed unknown, but a fixed estimate $\hat{\mathbf{I}}$ is available satisfying $\|\tilde{\mathbf{I}}\| \leq p$, where $p > 0$ is a known bound and $\tilde{\mathbf{I}} \stackrel{\Delta}{=} \hat{\mathbf{I}} - \mathbf{I}$ is the inertia matrix estimate error. We also assume bounds $\underline{\lambda}_I \leq \lambda_{min}(\mathbf{I})$ and $\overline{\lambda}_I \geq \lambda_{max}(\mathbf{I})$ are available, and that the disturbance torque has a known bound, $\|\boldsymbol{\tau}_d\| \leq \bar{\tau}_d$.

The desired inertial attitude trajectory is denoted $\mathbf{C}_d(t)$ in terms of the rotation matrix. The desired angular velocity $\boldsymbol{\omega}_d(t)$ (expressed in desired body coordinates) satisfies $\dot{\mathbf{C}}_d(t) = -\boldsymbol{\omega}_d^{\times}(t)\mathbf{C}_d(t)$ (see [10, p. 31]). It is assumed that $\boldsymbol{\omega}_d(t)$ and $\dot{\boldsymbol{\omega}}_d(t)$ are continuous and bounded by $\|\boldsymbol{\omega}_d(t)\| \leq \bar{w}_d$ and $\|\dot{\boldsymbol{\omega}}_d(t)\| \leq \dot{\bar{w}}_d$ for all $t \in \mathbb{R}^+$. Given the true spacecraft inertial attitude $\mathbf{C}(t)$, the attitude tracking error is defined as the true attitude relative to the desired attitude,

$$\delta\mathbf{C}(t) \stackrel{\Delta}{=} \mathbf{C}(t)\mathbf{C}_d^T(t). \tag{4}$$

When expressed in the true body coordinates, the angular velocity of the true spacecraft body frame with respect to the desired spacecraft body frame is

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \delta\mathbf{C}\boldsymbol{\omega}_d. \tag{5}$$

Let $(\mathbf{q}, q_4)$ be a quaternion parameterization of the attitude tracking error $\delta\mathbf{C}$. That is [10, p. 30],

$$\delta\mathbf{C}(\mathbf{q}, q_4) = (q_4^2 - \mathbf{q}^T\mathbf{q})\mathbf{1} + 2\mathbf{q}\mathbf{q}^T - 2q_4\mathbf{q}^\times. \tag{6}$$

Then, the associated quaternion kinematics are given by [10, p. 31]

$$\begin{aligned} \dot{\mathbf{q}} &= \tfrac{1}{2}(\mathbf{q}^\times + q_4\mathbf{1})\delta\boldsymbol{\omega}, \\ \dot{q}_4 &= -\tfrac{1}{2}\mathbf{q}^T\delta\boldsymbol{\omega}. \end{aligned} \tag{7}$$

Equivalently, the rotation matrix kinematics satisfy

$$\delta\dot{\mathbf{C}} = -\delta\boldsymbol{\omega}^\times\delta\mathbf{C}. \tag{8}$$

It will be useful to define an auxiliary desired angular velocity, given by

$$\bar{\boldsymbol{\omega}}_d \triangleq \delta\mathbf{C}(\mathbf{q}, q_4)\boldsymbol{\omega}_d - \boldsymbol{\Lambda}\mathbf{q}, \tag{9}$$

with $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T > \mathbf{0}$ some constant positive-definite matrix. Making use of (7), (8) and (9), we have

$$\dot{\bar{\boldsymbol{\omega}}}_d = -\delta\boldsymbol{\omega}^\times\delta\mathbf{C}(\mathbf{q}, q_4)\boldsymbol{\omega}_d + \delta\mathbf{C}(\mathbf{q}, q_4)\dot{\boldsymbol{\omega}}_d - \frac{\boldsymbol{\Lambda}}{2}(\mathbf{q}^\times + q_4\mathbf{1})\delta\boldsymbol{\omega}. \tag{10}$$

Finally, the filtered angular velocity error is defined as

$$\tilde{\boldsymbol{\omega}} \triangleq \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_d = \delta\boldsymbol{\omega} + \boldsymbol{\Lambda}\mathbf{q}. \tag{11}$$

3.1 Measurement models

We assume attitude tracking error measurements of the form

$$\begin{aligned} \mathbf{q}^m &= \mathbf{q} + \left((v_{q4} - 1)\mathbf{q} + q_4\mathbf{v}_q + \mathbf{q}^\times\mathbf{v}_q\right), \\ q_4^m &= q_4 + \left((v_{q4} - 1)q_4 - \mathbf{q}^T\mathbf{v}_q\right), \end{aligned} \tag{12}$$

where $(\mathbf{v}_q, v_{q4})$ is a quaternion representation of the attitude measurement error, with

$$\|\mathbf{v}_q\| \leq \bar{v}_q < 1. \tag{13}$$

Note that $(\mathbf{v}_q, v_{q4})$ represents a rotational transformation from the true body-frame to the measured body frame. Hence, the measured attitude tracking error in (12) is given by a quaternion product between the true attitude tracking error and the measurement error [10, p. 17]. Now, both $(\mathbf{q}^m, q_4^m)$ and $(-\mathbf{q}^m, -q_4^m)$ represent the same measured attitude tracking error, and both $(\mathbf{v}_q, v_{q4})$ and $(-\mathbf{v}_q, -v_{q4})$ represent the same measurement error. Using (12) and performing some algebra (making use of the unit norm constraint on $(\mathbf{v}_q, v_{q4})$ and $(\mathbf{v}_q, v_{q4})$), we obtain

$$\| (\mathbf{q}^m - \mathbf{q}, q_4^m - q_4) \| = \sqrt{2\,(1 - v_{q4})}. \tag{14}$$

This clearly shows that $(\mathbf{q}^m, q_4^m)$ is closest to $(\mathbf{q}, q_4)$ (in the norm sense), and $(-\mathbf{q}^m, -q_4^m)$ is closest to $(-\mathbf{q}, -q_4)$ when $v_{q4} > 0$. As explained in Remark 1, the attitude control can use either $(\mathbf{q}, q_4)$ or $(-\mathbf{q}, -q_4)$, provided no switch is made from one to the other. The above analysis shows that this can be assured by preventing switching between $(\mathbf{q}^m, q_4^m)$ and $(-\mathbf{q}^m, -q_4^m)$. Finally, it also allows us to set $v_{q4} > 0$ in (12) in all subsequent error analysis.

Next, we assume body-rate measurements of the form

$$\boldsymbol{\omega}^m = \boldsymbol{\omega} + \mathbf{v}_\omega, \tag{15}$$

where the body-rate measurement error satisfies

$$\|\mathbf{v}_\omega\| \leq \bar{v}_\omega. \tag{16}$$

The measurement error models in (12) and (15) lead to measured versions of the quantities in equations (5), (9), (10) and (11), given by

$$\begin{aligned}
\bar{\boldsymbol{\omega}}_d^m &= \delta\mathbf{C}^m\boldsymbol{\omega}_d - \boldsymbol{\Lambda}\mathbf{q}^m, \\
\tilde{\boldsymbol{\omega}}^m &= \boldsymbol{\omega}^m - \bar{\boldsymbol{\omega}}_d^m, \\
\dot{\bar{\boldsymbol{\omega}}}_d^m &= -\delta\boldsymbol{\omega}^{m\times}\delta\mathbf{C}^m\boldsymbol{\omega}_d + \delta\mathbf{C}^m\dot{\boldsymbol{\omega}}_d - \frac{\boldsymbol{\Lambda}}{2}\left(\mathbf{q}^{m\times} + q_4^m\mathbf{1}\right)\delta\boldsymbol{\omega}^m, \\
\delta\boldsymbol{\omega}^m &= \boldsymbol{\omega}^m - \delta\mathbf{C}^m\boldsymbol{\omega}_d,
\end{aligned} \tag{17}$$

where $\delta\mathbf{C}^m = \mathbf{C}(\mathbf{v}_q, v_{q4})\delta\mathbf{C}(\mathbf{q}, q_4)$ (see (12) for the quaternion equivalent). Making use of (12) and (15), the measured quantities in (17) can be rewritten as

$$\begin{aligned}
\bar{\boldsymbol{\omega}}_d^m &= \bar{\boldsymbol{\omega}}_d + \mathbf{v}_{\bar{\omega}_d}, \\
\tilde{\boldsymbol{\omega}}^m &= \tilde{\boldsymbol{\omega}} + \mathbf{v}_{\tilde{\omega}}, \\
\delta\boldsymbol{\omega}^m &= \delta\boldsymbol{\omega} + \mathbf{v}_{\delta\omega}, \\
\dot{\bar{\boldsymbol{\omega}}}_d^m &= \dot{\bar{\boldsymbol{\omega}}}_d + \mathbf{v}_{\dot{\bar{\omega}}_d},
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\mathbf{v}_{\bar{\omega}_d} &= 2\left[\mathbf{v}_q^\times\mathbf{v}_q^\times - v_{q4}\mathbf{v}_q^\times\right]\delta\mathbf{C}\boldsymbol{\omega}_d - \boldsymbol{\Lambda}\left[(v_{q4}-1)\mathbf{q} + q_4\mathbf{v}_q + \mathbf{q}^\times\mathbf{v}_q\right], \\
\mathbf{v}_{\tilde{\omega}} &= \mathbf{v}_\omega - \mathbf{v}_{\bar{\omega}_d}, \\
\mathbf{v}_{\delta\omega} &= \mathbf{v}_\omega - 2\left[\mathbf{v}_q^\times\mathbf{v}_q^\times - v_{q4}\mathbf{v}_q^\times\right]\delta\mathbf{C}\boldsymbol{\omega}_d, \\
\mathbf{v}_{\dot{\bar{\omega}}_d} &= -\mathbf{v}_{\delta\omega}^\times\delta\mathbf{C}^m\boldsymbol{\omega}_d - \delta\boldsymbol{\omega}^\times 2\left[\mathbf{v}_q^\times\mathbf{v}_q^\times - v_{q4}\mathbf{v}_q^\times\right]\delta\mathbf{C}\boldsymbol{\omega}_d + 2\left[\mathbf{v}_q^\times\mathbf{v}_q^\times - v_{q4}\mathbf{v}_q^\times\right]\delta\mathbf{C}\dot{\boldsymbol{\omega}}_d \\
&\quad -\frac{\boldsymbol{\Lambda}}{2}(v_{q4}-1)\left[\mathbf{q}^\times + q_4\mathbf{1}\right]\delta\boldsymbol{\omega} - \frac{\boldsymbol{\Lambda}}{2}\left[q_4\mathbf{v}_q^\times + (\mathbf{q}^\times\mathbf{v}_q)^\times - \mathbf{q}^T\mathbf{v}_q\mathbf{1}\right]\delta\boldsymbol{\omega} - \frac{\boldsymbol{\Lambda}}{2}\left[\mathbf{q}^{m\times} + q_4^m\mathbf{1}\right]\mathbf{v}_{\delta\omega}.
\end{aligned} \tag{19}$$

The quantities in (19) can be bounded by

$$\|\mathbf{v}_{\bar{\omega}_d}\| \le \bar{v}_{\bar{\omega}_d}, \;\; \|\mathbf{v}_{\tilde{\omega}}\| \le \bar{v}_{\tilde{\omega}}, \;\; \|\mathbf{v}_{\delta\omega}\| \le \bar{v}_{\delta\omega}, \;\; \|\mathbf{v}_{\dot{\bar{\omega}}_d}\| \le \bar{v}_{\dot{\bar{\omega}}_d}, \tag{20}$$

where

$$\begin{aligned}
\bar{v}_{\bar{\omega}_d}(\|\mathbf{q}\|) &= \|\mathbf{q}\|\lambda_{max}(\boldsymbol{\Lambda})\left[1 - \sqrt{1-\bar{v}_q^2} + \bar{v}_q\right] + \left[2(\bar{v}_q+1)\bar{w}_d + \lambda_{max}(\boldsymbol{\Lambda})\right]\bar{v}_q, \\
\bar{v}_{\tilde{\omega}}(\|\mathbf{q}\|) &= \bar{v}_\omega + \bar{v}_{\bar{\omega}_d}(\|\mathbf{q}\|), \\
\bar{v}_{\delta\omega} &= \bar{v}_\omega + 2\bar{v}_q(\bar{v}_q+1)\bar{w}_d, \\
\bar{v}_{\dot{\bar{\omega}}_d}(\|\mathbf{q}\|) &= \left[\|\tilde{\boldsymbol{\omega}}\| + \lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\|\right]\left[2\bar{v}_q(\bar{v}_q+1)\bar{w}_d + \frac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\left(1 - \sqrt{1-\bar{v}_q^2} + \bar{v}_q(1+2\|\mathbf{q}\|)\right)\right] \\
&\quad +\bar{v}_{\delta\omega}\left[\bar{w}_d + \frac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\right] + 2\bar{v}_q(\bar{v}_q+1)\dot{\bar{w}}_d.
\end{aligned} \tag{21}$$

4 Non-adaptive control

Consider the control law

$$\boldsymbol{\tau}_c = \hat{\mathbf{I}}\dot{\bar{\boldsymbol{\omega}}}_d^m + \bar{\boldsymbol{\omega}}_d^{m\times}\hat{\mathbf{I}}\boldsymbol{\omega}^m - \mathbf{K}\tilde{\boldsymbol{\omega}}^m, \tag{22}$$

where $\mathbf{K} = \mathbf{K}^T > 0$ is a positive definite gain matrix. This may be rewritten as

$$\boldsymbol{\tau}_c = \mathbf{I}\dot{\bar{\boldsymbol{\omega}}}_d + \bar{\boldsymbol{\omega}}_d^\times\mathbf{I}\boldsymbol{\omega} - \mathbf{K}\tilde{\boldsymbol{\omega}} + \mathbf{a}, \tag{23}$$

where

$$\begin{aligned}
\mathbf{a} &= \tilde{\mathbf{I}}\dot{\bar{\boldsymbol{\omega}}}_d + \bar{\boldsymbol{\omega}}_d^\times\tilde{\mathbf{I}}\bar{\boldsymbol{\omega}}_d + \bar{\boldsymbol{\omega}}_d^\times\tilde{\mathbf{I}}\tilde{\boldsymbol{\omega}} \\
&\quad +\hat{\mathbf{I}}\mathbf{v}_{\dot{\bar{\omega}}_d} + \mathbf{v}_{\bar{\omega}_d}^\times\hat{\mathbf{I}}\left(\tilde{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}}_d\right) + \left(\bar{\boldsymbol{\omega}}_d + \mathbf{v}_{\bar{\omega}_d}\right)^\times\hat{\mathbf{I}}\mathbf{v}_\omega - \mathbf{K}\mathbf{v}_{\tilde{\omega}}.
\end{aligned} \tag{24}$$

Substituting (23) into (3) and rearranging, we obtain

$$\mathbf{I}\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^\times\mathbf{I}\boldsymbol{\omega} = -\mathbf{K}\tilde{\boldsymbol{\omega}} + \boldsymbol{\tau}_d + \mathbf{a}. \tag{25}$$

We now present an iterative algorithm for finding an ultimate upper bound on $\|\mathbf{q}(t)\|$ and $\|\delta\boldsymbol{\omega}(t)\|$ for the closed-loop system (25).

Algorithm 1

Set $\bar{q}_0 = 1$. For $i = 1, ..., n$, where n is some (user determined) finite positive integer, compute

$$\begin{aligned}
\bar{v}^i_{\tilde{\omega}_d} &= \bar{q}_{i-1}\lambda_{max}(\boldsymbol{\Lambda})\left[1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q\right]+\left[2(\bar{v}_q+1)+\lambda_{max}(\boldsymbol{\Lambda})\right]\bar{v}_q, \\
\bar{v}^i_{\tilde{\omega}} &= \bar{v}_\omega + \bar{v}^i_{\tilde{\omega}_d}, \\
\bar{v}_{\delta\omega} &= \bar{v}_\omega + 2\bar{v}_q(\bar{v}_q+1)\bar{w}_d, \\
c_i &= p\left(2\bar{w}_d+\lambda_{max}(\boldsymbol{\Lambda})\left[\tfrac{1}{2}+\bar{q}_{i-1}\right]\right) \\
&\quad +\|\hat{\mathbf{I}}\|\left(2\bar{v}_q(\bar{v}_q+1)\bar{w}_d+\tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\left(1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q(1+2\bar{q}_{i-1})\right)+\bar{v}^i_{\tilde{\omega}_d}\right), \\
d_i &= p\left(\dot{\bar{w}}_d+\lambda_{max}(\boldsymbol{\Lambda})\left[\bar{w}_d+\tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\right]\bar{q}_{i-1}+\left[\bar{w}_d+\lambda_{max}(\boldsymbol{\Lambda})\bar{q}_{i-1}\right]^2\right) \\
&\quad +\|\hat{\mathbf{I}}\|\left(\lambda_{max}(\boldsymbol{\Lambda})\bar{q}_{i-1}\left[2\bar{v}_q(\bar{v}_q+1)\bar{w}_d+\tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\left(1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q(1+2\bar{q}_{i-1})\right)\right]\right. \\
&\quad +\bar{v}_{\delta\omega}\left[\bar{w}_d+\tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\right]+2\bar{v}_q(\bar{v}_q+1)\dot{\bar{w}}_d+\bar{v}^i_{\tilde{\omega}_d}\left[\bar{w}_d+\lambda_{max}(\boldsymbol{\Lambda})\bar{q}_{i-1}+\bar{v}_\omega\right] \\
&\quad \left.+\bar{v}_\omega\left[\bar{w}_d+\lambda_{max}(\boldsymbol{\Lambda})\bar{q}_{i-1}\right]\right)+\lambda_{max}(\mathbf{K})\bar{v}^i_{\tilde{\omega}}. \\
\bar{r}_i &= \tfrac{d_i+\bar{\tau}_d}{\lambda_{min}(\mathbf{K})-c_i}\sqrt{\tfrac{\overline{\lambda}_I}{\underline{\lambda}_I}}, \\
\bar{q}_i &= \tfrac{\bar{r}_i}{\lambda_{min}(\boldsymbol{\Lambda})}.
\end{aligned} \tag{26}$$

Theorem 1

Assume that $0 \le \bar{q}_1 < 1$ and $\lambda_{min}(\mathbf{K}) > c_1$. Then, the sequences $\{\bar{r}_i\}$ and $\{\bar{q}_i\}$ in Algorithm 1 are decreasing and convergent. Furthermore, for the system (3) with control law (22),

$$\begin{aligned}
\limsup_{t\to\infty}\|\mathbf{q}(t)\| &\le \bar{q}_i, \\
\limsup_{t\to\infty}\|\delta\boldsymbol{\omega}(t)\| &\le \left(\frac{\lambda_{max}(\boldsymbol{\Lambda})}{\lambda_{min}(\boldsymbol{\Lambda})}+1\right)\bar{r}_i,
\end{aligned}$$

for $i \ge 1$.

Proof

Suppose for some i that $0 \le \bar{q}_{i-1} < \bar{q}_{i-2}$ and $\lambda_{min}(\mathbf{K}) > c_{i-1}$. This condition holds for $i = 2$ by hypothesis. Then, from (26), it is clear that $0 \le c_i < c_{i-1}$ and $0 \le d_i < d_{i-1}$. Consequently, $\lambda_{min}(\mathbf{K}) > c_i$ also, and we can conclude that

$$0 \le \bar{q}_i = \frac{d_i+\bar{\tau}_d}{\lambda_{min}(\boldsymbol{\Lambda})\left(\lambda_{min}(\mathbf{K})-c_i\right)}\sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}} < \frac{d_{i-1}+\bar{\tau}_d}{\lambda_{min}(\boldsymbol{\Lambda})\left(\lambda_{min}(\mathbf{K})-c_{i-1}\right)}\sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}} = \bar{q}_{i-1}.$$

Therefore, by induction we conclude that $\{\bar{q}_i\}$ is decreasing and bounded. Hence, it is convergent. Since $\bar{r}_i = \lambda_{min}(\boldsymbol{\Lambda})\bar{q}_i$, it follows that $\{\bar{r}_i\}$ is decreasing and convergent also.

Now, consider the Lyapunov-like function $V(\tilde{\boldsymbol{\omega}}) = \frac{1}{2}\tilde{\boldsymbol{\omega}}^T\mathbf{I}\tilde{\boldsymbol{\omega}}$. Taking the derivative along a trajectory of (25), we have

$$\dot{V} = -\tilde{\boldsymbol{\omega}}^T \mathbf{K} \tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^T \boldsymbol{\tau}_d + \tilde{\boldsymbol{\omega}}^T \mathbf{a}. \tag{27}$$

Let us now find an upper bound for $\|\mathbf{a}\|$. Since $\delta\mathbf{C}$ is orthonormal [10, pp. 9–10], we obtain from (9),

$$\begin{aligned} \|\bar{\boldsymbol{\omega}}_d\| &\leq \|\boldsymbol{\omega}_d\| + \lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\|, \\ &\leq \bar{w}_d + \lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\|. \end{aligned} \tag{28}$$

Differentiating (9) we obtain

$$\dot{\bar{\boldsymbol{\omega}}}_d = \delta\dot{\mathbf{C}}\boldsymbol{\omega}_d + \delta\mathbf{C}\dot{\boldsymbol{\omega}}_d - \boldsymbol{\Lambda}\dot{\mathbf{q}}.$$

Making use of (7), (8) and (11), it follows that

$$\dot{\bar{\boldsymbol{\omega}}}_d = (\delta\mathbf{C}\boldsymbol{\omega}_d)^\times \tilde{\boldsymbol{\omega}} - (\delta\mathbf{C}\boldsymbol{\omega}_d)^\times \boldsymbol{\Lambda}\mathbf{q} + \delta\mathbf{C}\dot{\boldsymbol{\omega}}_d - \frac{\boldsymbol{\Lambda}}{2}\left(\mathbf{q}^\times + q_4\mathbf{1}\right)\tilde{\boldsymbol{\omega}} + \frac{\boldsymbol{\Lambda}}{2}\left(\mathbf{q}^\times + q_4\mathbf{1}\right)\boldsymbol{\Lambda}\mathbf{q}.$$

Upper bounding this we obtain

$$\|\dot{\bar{\boldsymbol{\omega}}}_d\| \leq \dot{\bar{w}}_d + \left(\bar{w}_d + \frac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\right)\|\tilde{\boldsymbol{\omega}}\| + \lambda_{max}(\boldsymbol{\Lambda})\left(\bar{w}_d + \frac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\right)\|\mathbf{q}\|, \tag{29}$$

where we have made use of the orthonormality of $\delta\mathbf{C}$ and the facts that $\|\mathbf{a}^\times\| = \|\mathbf{a}\|$ for any $\mathbf{a} \in R^3$ and $\|\mathbf{q}^\times + q_4\mathbf{1}\| = 1$. Making use of (28), (29), (20) and (21), $\mathbf{a}$ may be bounded by

$$\|\mathbf{a}\| \leq \|\tilde{\boldsymbol{\omega}}\|c + d, \tag{30}$$

where

$$\begin{aligned} c(\|\mathbf{q}\|) &= p\left(2\bar{w}_d + \lambda_{max}(\boldsymbol{\Lambda})\left[\tfrac{1}{2} + \|\mathbf{q}\|\right]\right) \\ &\quad + \|\hat{\mathbf{I}}\|\left(2\bar{v}_q(\bar{v}_q + 1)\bar{w}_d + \tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\left(1 - \sqrt{1 - \bar{v}_q^2} + \bar{v}_q(1 + 2\|\mathbf{q}\|)\right) + \bar{v}_{\bar{\omega}_d}\right), \end{aligned}$$

$$\begin{aligned} d(\|\mathbf{q}\|) &= p\left(\dot{\bar{w}}_d + \lambda_{max}(\boldsymbol{\Lambda})\left[\bar{w}_d + \tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\right]\|\mathbf{q}\| + \left[\bar{w}_d + \lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\|\right]^2\right) \\ &\quad + \|\hat{\mathbf{I}}\|\left(\lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\|\left[2\bar{v}_q(\bar{v}_q + 1)\bar{w}_d + \tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\left(1 - \sqrt{1 - \bar{v}_q^2} + \bar{v}_q(1 + 2\|\mathbf{q}\|)\right)\right]\right. \\ &\quad + \bar{v}_{\delta\omega}\left[\bar{w}_d + \tfrac{\lambda_{max}(\boldsymbol{\Lambda})}{2}\right] + 2\bar{v}_q(\bar{v}_q + 1)\dot{\bar{w}}_d + \bar{v}_{\bar{\omega}_d}\left[\bar{w}_d + \lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\| + \bar{v}_\omega\right] \\ &\quad \left. + \bar{v}_\omega\left[\bar{w}_d + \lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\|\right]\right) + \lambda_{max}(\mathbf{K})\bar{v}_{\tilde{\omega}}. \end{aligned}$$

Making use of (30) and the bound $\|\boldsymbol{\tau}_d\| \leq \bar{\tau}_d$, (27) is upper bounded by

$$\dot{V} \leq -\|\tilde{\boldsymbol{\omega}}\|\left[(\lambda_{min}(\mathbf{K}) - c)\|\tilde{\boldsymbol{\omega}}\| - (d + \bar{\tau}_d)\right]. \tag{31}$$

Suppose now that for some i, $\lambda_{min}(\mathbf{K}) > c_i$ and for any $\epsilon_1 > 0$, there exists a $T_{i-1}(\epsilon_1) \geq 0$ such that for all

$t \geq T_{i-1}(\epsilon_1)$, $\|\mathbf{q}(t)\| \leq \bar{q}_{i-1} + \epsilon_1$. This condition clearly holds for $i = 1$ with $T_{i-1}(\epsilon_1) = 0$. We then find from (31) that

$$\dot{V} \leq -\|\tilde{\boldsymbol{\omega}}\| \left[(\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)) \|\tilde{\boldsymbol{\omega}}\| - \left(\bar{d}_i(\epsilon_1) + \bar{\tau}_d \right) \right], \ \ \forall t \geq T_{i-1}(\epsilon_1), \tag{32}$$

where

$$\begin{aligned} \bar{c}_i(\epsilon_1) &= c(\bar{q}_{i-1} + \epsilon_1), \\ \bar{d}_i(\epsilon_1) &= d(\bar{q}_{i-1} + \epsilon_1). \end{aligned} \tag{33}$$

Note that $c_i = \bar{c}_i(0)$ and $d_i = \bar{d}_i(0)$. Now, select $\epsilon_1 > 0$ small enough such that $\lambda_{min}(\mathbf{K}) > \bar{c}_i(\epsilon_1)$. Then, from (32), for any $\epsilon_2 > 0$, $\dot{V} \leq -\left(\frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} + \epsilon_2 \right) (\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)) \, \epsilon_2 < 0$ outside the set

$$F_{\epsilon_2} \triangleq \left\{ \tilde{\boldsymbol{\omega}} : \|\tilde{\boldsymbol{\omega}}\| \leq \frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} + \epsilon_2 \right\},$$

for all $t \geq T_{i-1}(\epsilon_1)$. Now, $V(\tilde{\boldsymbol{\omega}})$ can be bounded by $\underline{\lambda}_I \|\tilde{\boldsymbol{\omega}}\|^2 \leq 2V(\tilde{\boldsymbol{\omega}}) \leq \overline{\lambda}_I \|\tilde{\boldsymbol{\omega}}\|^2$. Therefore, the compact set

$$G_{\epsilon_2} \triangleq \left\{ \tilde{\boldsymbol{\omega}} : 2V(\tilde{\boldsymbol{\omega}}) \leq \overline{\lambda}_I \left(\frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} + \epsilon_2 \right)^2 \right\}$$

contains F_{ϵ_2}, and $\dot{V} \leq -\left(\frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} + \epsilon_2 \right) (\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)) \, \epsilon_2 < 0$ outside G_{ϵ_2} for all $t \geq T_{i-1}(\epsilon_1)$. Consequently, there exists a $T_i'(\epsilon_1, \epsilon_2) \geq T_{i-1}(\epsilon_1)$ such that $\tilde{\boldsymbol{\omega}}(t) \in G_{\epsilon_2}$ for all $t \geq T_i'(\epsilon_1, \epsilon_2)$. The set

$$H_{\epsilon_2} \triangleq \left\{ \tilde{\boldsymbol{\omega}} : \|\tilde{\boldsymbol{\omega}}\|^2 \leq \frac{\overline{\lambda}_I}{\underline{\lambda}_I} \left(\frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} + \epsilon_2 \right)^2 \right\},$$

contains G_{ϵ_2}. Therefore, $\|\tilde{\boldsymbol{\omega}}(t)\| \leq \frac{\bar{d}_i + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i} \sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}} + \bar{\epsilon}_2$ for all $t \geq T_i'(\epsilon_1, \epsilon_2)$, where $\bar{\epsilon}_2 = \sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}} \epsilon_2$. Now, it can readily be verified that $\frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} \sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}}$ is continuous in ϵ_1 and is strictly increasing for $\epsilon_1 \geq 0$, with $\frac{\bar{d}_i(0) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(0)} \sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}} = \bar{r}_i$ and $\lim_{\bar{c}_i(\epsilon_1) \uparrow \lambda_{min}(\mathbf{K})} \frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} \sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}} = \infty$. Therefore, given any $\epsilon_3 > 0$, it is possible to find $\epsilon_1 > 0$ (small enough such that $\lambda_{min}(\mathbf{K}) > \bar{c}_i(\epsilon_1)$) and $\epsilon_2 > 0$ such that $\bar{r}_i + \epsilon_3 = \frac{\bar{d}_i(\epsilon_1) + \bar{\tau}_d}{\lambda_{min}(\mathbf{K}) - \bar{c}_i(\epsilon_1)} \sqrt{\frac{\overline{\lambda}_I}{\underline{\lambda}_I}} + \bar{\epsilon}_2$. We conclude that for any $\epsilon_3 > 0$, there exists a $T_i'(\epsilon_3) \geq T_{i-1}(\epsilon_1)$ such that

$$\|\tilde{\boldsymbol{\omega}}(t)\| \leq \bar{r}_i + \epsilon_3, \ \ \forall t \geq T_i'(\epsilon_3).$$

Consequently

$$\limsup_{t \to \infty} \|\tilde{\boldsymbol{\omega}}(t)\| \leq \bar{r}_i.$$

From Lemma 1, given $\epsilon_3 > 0$, for any $\epsilon_4 > 0$, there exists a $T_i(\epsilon_3, \epsilon_4) \geq T_i'(\epsilon_3)$ such that $\|\mathbf{q}(t)\| \leq \frac{\bar{r}_i + \epsilon_3}{\lambda_{min}(\boldsymbol{\Lambda})} + \epsilon_4$ for all $t \geq T_i(\epsilon_3, \epsilon_4)$. Clearly, for any $\epsilon_5 > 0$, it is possible to find $\epsilon_3 > 0$ and $\epsilon_4 > 0$ such that $\bar{q}_i + \epsilon_5 = \frac{\bar{r}_i + \epsilon_3}{\lambda_{min}(\boldsymbol{\Lambda})} + \epsilon_4$.

Therefore, we conclude that for any $\epsilon_5 > 0$, there exists a $T_i(\epsilon_5) \geq T_{i-1}(\epsilon_1)$ such that

$$\|\mathbf{q}(t)\| \leq \bar{q}_i + \epsilon_5, \ \ \forall t \geq T_i(\epsilon_5).$$

Therefore, by induction we conclude that

$$\limsup_{t\to\infty} \|\mathbf{q}(t)\| \leq \bar{q}_i,$$

for all $i \geq 1$. Finally, since $\|\delta\boldsymbol{\omega}\| \leq \|\tilde{\boldsymbol{\omega}}\| + \lambda_{max}(\mathbf{\Lambda})\|\mathbf{q}\|$, the conclusion for $\|\delta\boldsymbol{\omega}(t)\|$ follows. □

Corollary 2

Let the conditions of Theorem 1 hold, and let $\bar{q} = \lim_{i\to\infty} \bar{q}_i$ and $\bar{r} = \lim_{i\to\infty} \bar{r}_i$. Then,

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\leq \bar{q}, \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\leq \left(\frac{\lambda_{max}(\mathbf{\Lambda})}{\lambda_{min}(\mathbf{\Lambda})} + 1\right)\bar{r}. \end{aligned}$$

Proof

Noting that $\bar{q} = \inf \bar{q}_i$ and $\bar{r} = \inf \bar{r}_i$, the conclusion follows from Theorem 1. □

5 Adaptive Control

We shall now present an adaptive controller. First, we rewrite the attitude dynamics (3) as

$$\dot{\boldsymbol{\omega}} = -\mathbf{P}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{P}\boldsymbol{\tau}_c + \boldsymbol{\tau}_{dP}, \tag{34}$$

where $\mathbf{P} = \mathbf{I}^{-1}$ and $\boldsymbol{\tau}_{dp} = \mathbf{P}\boldsymbol{\tau}_d$. We automatically have the bounds $\frac{1}{\bar{\lambda}_I} \leq \lambda_{min}(\mathbf{P})$, $\lambda_{max}(\mathbf{P}) \leq \frac{1}{\underline{\lambda}_I}$ and $\|\boldsymbol{\tau}_{dP}\| \leq \bar{\tau}_d/\underline{\lambda}_I$.

It can readily be shown by direct expansion that there exists a $\boldsymbol{\theta} \in \mathbb{R}^{18}$ (a function of $\mathbf{P}$ and $\mathbf{I}$) such that

$$\mathbf{\Phi}(\boldsymbol{\omega})\boldsymbol{\theta} = \mathbf{P}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega}, \tag{35}$$

where

$$\mathbf{\Phi}(\boldsymbol{\omega}) = \mathrm{diag}\left\{\boldsymbol{\phi}(\boldsymbol{\omega}), \boldsymbol{\phi}(\boldsymbol{\omega}), \boldsymbol{\phi}(\boldsymbol{\omega})\right\}, \tag{36}$$

and

$$\boldsymbol{\phi}(\boldsymbol{\omega}) = \begin{bmatrix} \omega_x^2 & \omega_y^2 & \omega_z^2 & \omega_x\omega_y & \omega_x\omega_z & \omega_y\omega_z \end{bmatrix}. \tag{37}$$

Therefore, the attitude dynamics (34) become

$$\dot{\boldsymbol{\omega}} = -\boldsymbol{\Phi}(\boldsymbol{\omega})\boldsymbol{\theta} + \mathbf{P}\boldsymbol{\tau}_c + \boldsymbol{\tau}_{dP}. \tag{38}$$

The quantities $\boldsymbol{\theta}$ and $\mathbf{P}$ are unknown. We shall denote their estimates by $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{P}}$ respectively. Noting that $\mathbf{P}$ is symmetric, we only estimate the upper triangular part.

It can readily be shown that

$$\mathbf{P}\boldsymbol{\tau}_c = \boldsymbol{\Psi}(\boldsymbol{\tau}_c)\boldsymbol{\theta}_p \tag{39}$$

where

$$\boldsymbol{\Psi}(\boldsymbol{\tau}_c) = \begin{bmatrix} \tau_{cx} & 0 & 0 & \tau_{cy} & \tau_{cz} & 0 \\ 0 & \tau_{cy} & 0 & \tau_{cx} & 0 & \tau_{cz} \\ 0 & 0 & \tau_{cz} & 0 & \tau_{cx} & \tau_{cy} \end{bmatrix}, \quad \boldsymbol{\theta}_p = \begin{bmatrix} P_{11} \\ P_{22} \\ P_{33} \\ P_{12} \\ P_{13} \\ P_{23} \end{bmatrix},$$

and P_{ij} denotes the ij^{th} term of $\mathbf{P}$. As such, the attitude dynamics (34) may be rewritten as

$$\dot{\boldsymbol{\omega}} = -\boldsymbol{\Phi}(\boldsymbol{\omega})\boldsymbol{\theta} + \hat{\mathbf{P}}\boldsymbol{\tau}_c - \boldsymbol{\Psi}(\boldsymbol{\tau}_c)\tilde{\boldsymbol{\theta}}_p + \boldsymbol{\tau}_{dP}, \tag{40}$$

where $\tilde{\boldsymbol{\theta}}_p = \hat{\boldsymbol{\theta}}_p - \boldsymbol{\theta}_p$ and $\hat{\boldsymbol{\theta}}_p$ denotes the estimate of $\boldsymbol{\theta}_p$.

It is assumed that $\boldsymbol{\theta}^* \in \mathbb{R}^{18}$, $\boldsymbol{\theta}_p^* \in \mathbb{R}^6$, $\eta > 0$ and $\eta_p > 0$ are known such that $\boldsymbol{\theta} \in F \triangleq \left\{\mathbf{v} \in \mathbb{R}^{18} : \|\mathbf{v} - \boldsymbol{\theta}^*\| \le \eta\right\}$ and $\boldsymbol{\theta}_p \in F_p \triangleq \left\{\mathbf{v} \in \mathbb{R}^6 : \|\mathbf{v} - \boldsymbol{\theta}_p^*\| \le \eta_p\right\}$.

Consider the control and adaptation laws

$$\boldsymbol{\tau}_c = \hat{\mathbf{P}}^{-1}\left[\dot{\tilde{\boldsymbol{\omega}}}_d^m - \mathbf{K}\tilde{\boldsymbol{\omega}}^m + \boldsymbol{\Phi}(\boldsymbol{\omega}^m)\hat{\boldsymbol{\theta}}\right], \tag{41}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \begin{cases} \gamma \mathrm{Proj}_1\left(\hat{\boldsymbol{\theta}}, -\boldsymbol{\Phi}(\boldsymbol{\omega}^m)^T\tilde{\boldsymbol{\omega}}^m\right), & \|\tilde{\boldsymbol{\omega}}^m\| > \Omega_m, \\ \mathbf{0}, & \|\tilde{\boldsymbol{\omega}}^m\| \le \Omega_m, \end{cases} \quad \hat{\boldsymbol{\theta}}(0) = \boldsymbol{\theta}^*, \tag{42}$$

$$\dot{\hat{\boldsymbol{\theta}}}_p = \begin{cases} \gamma_p \mathrm{Proj}_2\left(\hat{\boldsymbol{\theta}}_p, \boldsymbol{\Psi}(\boldsymbol{\tau}_c)^T\tilde{\boldsymbol{\omega}}^m\right), & \|\tilde{\boldsymbol{\omega}}^m\| > \Omega_m, \\ \mathbf{0}, & \|\tilde{\boldsymbol{\omega}}^m\| \le \Omega_m, \end{cases} \quad \hat{\boldsymbol{\theta}}_p(0) = \boldsymbol{\theta}_p^*, \tag{43}$$

where $\mathbf{K} = \mathbf{K}^T > \mathbf{0}$ is a symmetric positive definite gain matrix, $\gamma, \gamma_p > 0$ are scalar positive gains, and $\Omega_m > 0$ is a deadzone size.

The projection operators in (42) and (43) are defined as [13]

$$\mathrm{Proj}_i(\boldsymbol{\upsilon}, \mathbf{y}) = \begin{cases} \mathbf{y} - \frac{d_i(\boldsymbol{\upsilon})\nabla d_i(\boldsymbol{\upsilon})\nabla d_i(\boldsymbol{\upsilon})^T\mathbf{y}}{||\nabla d_i(\boldsymbol{\upsilon})||_2^2}, & d_i(\boldsymbol{\upsilon}) > 0 \text{ and } \nabla d_i(\boldsymbol{\upsilon})^T\mathbf{y} > 0, \\ \mathbf{y}, & \text{otherwise}, \end{cases} \tag{44}$$

for $i = 1, 2$, where

$$d_1(\boldsymbol{\upsilon}) = \frac{||\boldsymbol{\upsilon} - \boldsymbol{\theta}^*||_2^2 - \eta^2}{\epsilon_{\upsilon 1}\eta^2}, \quad \epsilon_{\upsilon 1} > 0, \tag{45}$$

$$d_2(\boldsymbol{\upsilon}) = \frac{||\boldsymbol{\upsilon} - \boldsymbol{\theta}_p^*||_2^2 - \eta_p^2}{\epsilon_{\upsilon 2}\eta_p^2}, \quad \epsilon_{\upsilon 2} > 0. \tag{46}$$

The projection operator has two important properties [13]:

1. $$\left(\boldsymbol{\upsilon} - \bar{\boldsymbol{\theta}}\right)^T \left(\mathrm{Proj}_1(\boldsymbol{\upsilon}, \mathbf{y}) - \mathbf{y}\right) \leq \mathbf{0},$$

   for any $\boldsymbol{\upsilon} \in \mathbb{R}^{18}$, $\mathbf{y} \in \mathbb{R}^{18}$ and any $\bar{\boldsymbol{\theta}} \in F$, and

   $$\left(\boldsymbol{\upsilon}_p - \bar{\boldsymbol{\theta}}_p\right)^T \left(\mathrm{Proj}_2(\boldsymbol{\upsilon}_p, \mathbf{y}_p) - \mathbf{y}_p\right) \leq \mathbf{0},$$

   for any $\boldsymbol{\upsilon}_p \in \mathbb{R}^6$, $\mathbf{y}_p \in \mathbb{R}^6$ and any $\bar{\boldsymbol{\theta}}_p \in F_p$.

2. If $\hat{\boldsymbol{\theta}}(0) \in G \triangleq \left\{\mathbf{v} \in \mathbb{R}^{18} : \|\mathbf{v} - \boldsymbol{\theta}^*\| \leq \sqrt{1 + \epsilon_{\upsilon 1}}\eta\right\}$ and $\hat{\boldsymbol{\theta}}_p(0) \in G_p \triangleq \left\{\mathbf{v} \in \mathbb{R}^6 : \|\mathbf{v} - \boldsymbol{\theta}_p^*\| \leq \sqrt{1 + \epsilon_{\upsilon 2}}\eta_p\right\}$, then with the adaptation laws (42) and (43), $\hat{\boldsymbol{\theta}}(t) \in G$ and $\hat{\boldsymbol{\theta}}_p(t) \in G_p$ for all $t \geq 0$.

Property 2 can be used to guarantee that $\hat{\mathbf{P}}$ does not become singular, which is essential to implement (41). To see how, let $\mathbf{P}^*$ be the a-priori estimate of $\mathbf{P}$ corresponding to $\boldsymbol{\theta}_p^*$, and define $\Delta\hat{\boldsymbol{\theta}}_p \triangleq \hat{\boldsymbol{\theta}}_p - \boldsymbol{\theta}_p^*$. We require $\mathbf{P}^*$ to be diagonal. Note that since $\mathbf{P}^*$ is known, this can always be accomplished by a suitable transformation of coordinates of spacecraft attitude dynamics and the desired attitude trajectory. That is, let $\mathbf{C}_{p*}$ be the transformation of coordinates that renders $\mathbf{P}^*$ diagonal. Then, the desired attitude trajectory becomes $\mathbf{C}_{p*}\mathbf{C}_d(t)$ and the desired angular velocity correspondingly becomes $\mathbf{C}_{p*}\boldsymbol{\omega}_d(t)$. Accordingly, the estimate $\hat{\mathbf{P}}$ satisfies

$$\hat{\mathbf{P}} = \begin{bmatrix} P_{11}^* + \Delta\hat{P}_{11} & \Delta\hat{P}_{12} & \Delta\hat{P}_{13} \\ \Delta\hat{P}_{12} & P_{22}^* + \Delta\hat{P}_{22} & \Delta\hat{P}_{23} \\ \Delta\hat{P}_{13} & \Delta\hat{P}_{23} & P_{33}^* + \Delta\hat{P}_{33} \end{bmatrix}. \tag{47}$$

To proceed, we make use of Gershgorin's Theorem ([12, p. 161])

Gershgorin's Theorem

Every eigenvalue λ of an $n \times n$ matrix $\mathbf{A}$ satisfies at least one of the inequalities

$$|\lambda - A_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}|, \quad i = 1, ..., n. \tag{48}$$

Applying (48) to (47), and making use of the triangle inequality, we find that the eigenvalues λ of $\hat{\mathbf{P}}$ must satisfy one of

$$\begin{aligned}
|\lambda - P_{11}^*| &\leq |\Delta\hat{P}_{11}| + |\Delta\hat{P}_{12}| + |\Delta\hat{P}_{13}|, \\
|\lambda - P_{22}^*| &\leq |\Delta\hat{P}_{12}| + |\Delta\hat{P}_{22}| + |\Delta\hat{P}_{23}|, \\
|\lambda - P_{33}^*| &\leq |\Delta\hat{P}_{13}| + |\Delta\hat{P}_{23}| + |\Delta\hat{P}_{33}|.
\end{aligned} \tag{49}$$

The right-hand sides of each inequality can be upperbounded by $\|\Delta\hat{\boldsymbol{\theta}}_p\|_1$, where $\|\mathbf{x}\|_1 \triangleq \sum_{i=1}^{n} |x_i|$ is the standard absolute sum norm of $\mathbf{x} \in \mathbb{R}^n$. It is well-known that $\|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|$ for $\mathbf{x} \in \mathbb{R}^n$ (see for example [11, p. 88]). Therefore, we can upperbound (49) by

$$|\lambda - P_{ii}^*| \leq \sqrt{6}\|\Delta\hat{\boldsymbol{\theta}}_p\|, \;\; i = 1, 2, 3. \tag{50}$$

Each of the inequalities (50) define closed-intervals centered around P_{ii}^*, with radius $\sqrt{6}\|\Delta\hat{\boldsymbol{\theta}}_p\|$. To guarantee that $\hat{\mathbf{P}}$ stays positive definite, it is sufficient to ensure that none of these intervals contain the origin. This will be the case if

$$\sqrt{6}\|\Delta\hat{\boldsymbol{\theta}}_p\| < \min_{i=1,2,3} P_{ii}^*.$$

Finally, from Property 2 of the projection operator, a sufficient condition for $\hat{\mathbf{P}}$ to remain positive definite is

$$\sqrt{1 + \epsilon_{v2}}\eta_p < \frac{1}{\sqrt{6}} \min_{i=1,2,3} P_{ii}^*. \tag{51}$$

This places restrictions on ϵ_{v2} and η_p.

Making use of (18), and substituting (41) into (40), the closed-loop equation for the filtered error is

$$\dot{\tilde{\boldsymbol{\omega}}} = \boldsymbol{\Phi}(\boldsymbol{\omega}^m)\tilde{\boldsymbol{\theta}} - \mathbf{K}\tilde{\boldsymbol{\omega}}^m - \boldsymbol{\Psi}(\boldsymbol{\tau}_c)\tilde{\boldsymbol{\theta}}_p + [\boldsymbol{\Phi}(\boldsymbol{\omega}^m) - \boldsymbol{\Phi}(\boldsymbol{\omega})]\,\boldsymbol{\theta} + \mathbf{v}_{\dot{\omega}_d} + \boldsymbol{\tau}_{dP}. \tag{52}$$

where $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$. Following [14], we now make the additional assumption that the measurement errors $\mathbf{v}_q$ and $\mathbf{v}_\omega$ are differentiable, with bounds

$$\|\dot{\mathbf{v}}_q\| \leq \dot{\bar{v}}_q, \;\; \|\dot{\mathbf{v}}_\omega\| \leq \dot{\bar{v}}_\omega \tag{53}$$

It can be readily shown that (13) together with (53) lead to

$$|\dot{v}_{q4}| \leq \frac{\bar{v}_q \dot{\bar{v}}_q}{\sqrt{1-\bar{v}_q^2}} \triangleq \dot{\bar{v}}_{q4}. \tag{54}$$

Differentiating the expression for $\tilde{\boldsymbol{\omega}}^m$ in (18) and substituting into (52) yields

$$\dot{\tilde{\boldsymbol{\omega}}}^m = \boldsymbol{\Phi}(\boldsymbol{\omega}^m)\tilde{\boldsymbol{\theta}} - \mathbf{K}\tilde{\boldsymbol{\omega}}^m - \boldsymbol{\Psi}(\boldsymbol{\tau}_c)\tilde{\boldsymbol{\theta}}_p + \left[\boldsymbol{\Phi}(\boldsymbol{\omega}^m) - \boldsymbol{\Phi}(\boldsymbol{\omega})\right]\boldsymbol{\theta} + \mathbf{v}_{\dot{\bar{\omega}}_d} + \dot{\mathbf{v}}_{\tilde{\omega}} + \boldsymbol{\tau}_{dP}. \tag{55}$$

Now we shall find bounds for some of the terms on the right-hand side of (55). We start with the term $\left[\boldsymbol{\Phi}(\boldsymbol{\omega}^m) - \boldsymbol{\Phi}(\boldsymbol{\omega})\right]\boldsymbol{\theta}$. Since $\boldsymbol{\theta} \in F$,

$$\|\boldsymbol{\theta}\| \leq \|\boldsymbol{\theta}^*\| + \eta.$$

From (36),

$$\boldsymbol{\Phi}(\boldsymbol{\omega}^m) = \boldsymbol{\Phi}(\boldsymbol{\omega}) + \boldsymbol{\Phi}(\mathbf{v}_{\boldsymbol{\omega}}) + \mathrm{diag}\left\{\mathbf{b}, \mathbf{b}, \mathbf{b}\right\},$$

where

$$\mathbf{b} = \boldsymbol{\omega}^T\mathbf{D}, \;\; \mathbf{D} = \begin{bmatrix} 2v_{\omega x} & 0 & 0 & v_{\omega y} & v_{\omega z} & 0 \\ 0 & 2v_{\omega y} & 0 & v_{\omega x} & 0 & v_{\omega z} \\ 0 & 0 & 2v_{\omega z} & 0 & v_{\omega x} & v_{\omega y} \end{bmatrix}.$$

After some work, the following bound can be obtained

$$\|\left[\boldsymbol{\Phi}(\boldsymbol{\omega}^m) - \boldsymbol{\Phi}(\boldsymbol{\omega})\right]\boldsymbol{\theta}\| \leq a\|\tilde{\boldsymbol{\omega}}^m\| + b, \tag{56}$$

where

$$\begin{aligned} a &= \bar{D}\left[\|\boldsymbol{\theta}^*\| + \eta\right], \\ b(\|\mathbf{q}\|) &= \left[\bar{\phi}_v + \bar{D}\left(\bar{v}_{\tilde{\omega}}(\|\mathbf{q}\|) + \bar{w}_d + \lambda_{max}(\boldsymbol{\Lambda})\|\mathbf{q}\|\right)\right]\left[\|\boldsymbol{\theta}^*\| + \eta\right], \end{aligned}$$

and

$$\bar{\phi}_v = \max_{\|\mathbf{v}_q\| \leq \bar{v}_q} \|\boldsymbol{\phi}(\mathbf{v}_q)\|, \;\; \bar{D} = \max_{\|\mathbf{v}_q\| \leq \bar{v}_q} \|\mathbf{D}\|,$$

and $\boldsymbol{\phi}$ is given in (37). It can readily be shown that

$$\bar{\phi}_v = \bar{v}_q^2, \;\; \bar{D} \leq \sqrt{6}\bar{v}_q. \tag{57}$$

Next, we examine $\mathbf{v}_{\dot{\bar{\omega}}_d}$. From (20), (21) and (18), it is readily found that

$$\|\mathbf{v}_{\dot{\bar{\omega}}_d}\| \leq c\|\tilde{\boldsymbol{\omega}}^m\| + d, \tag{58}$$

where

$$
\begin{aligned}
c(\|\mathbf{q}\|) &= 2\bar{v}_q(\bar{v}_q+1)\bar{w}_d + \tfrac{\lambda_{max}(\mathbf{\Lambda})}{2}\left(1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q(1+2\|\mathbf{q}\|)\right), \\
d(\|\mathbf{q}\|) &= [\bar{v}_{\tilde{\omega}}(\|\mathbf{q}\|)+\lambda_{max}(\mathbf{\Lambda})\|\mathbf{q}\|]\left[2\bar{v}_q(\bar{v}_q+1)\bar{w}_d + \tfrac{\lambda_{max}(\mathbf{\Lambda})}{2}\left(1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q(1+2\|\mathbf{q}\|)\right)\right] \\
&\quad +\bar{v}_{\delta\omega}\left[\bar{w}_d+\tfrac{\lambda_{max}(\mathbf{\Lambda})}{2}\right]+2\bar{v}_q(\bar{v}_q+1)\dot{\bar{w}}_d.
\end{aligned}
$$

Finally, we treat $\dot{\mathbf{v}}_{\tilde{\omega}}$. Differentiating the expression for $\mathbf{v}_{\tilde{\omega}}$ in (19) using (7), and making use of the fact that $\delta\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}}^m - \mathbf{v}_{\tilde{\omega}} - \mathbf{\Lambda}\mathbf{q}$, the following bound can be obtained.

$$
\|\dot{\mathbf{v}}_{\tilde{\omega}}\| \le e\|\tilde{\boldsymbol{\omega}}^m\| + f, \tag{59}
$$

where

$$
\begin{aligned}
e(\|\mathbf{q}\|) &= 2\bar{v}_q(\bar{v}_q+1)\bar{w}_d + \tfrac{\lambda_{max}(\mathbf{\Lambda})}{2}\left[1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q(1+\|\mathbf{q}\|)\right], \\
f(\|\mathbf{q}\|) &= [\bar{v}_{\tilde{\omega}}(\|\mathbf{q}\|)+\lambda_{max}(\mathbf{\Lambda})\|\mathbf{q}\|]\left[2\bar{v}_q(\bar{v}_q+1)\bar{w}_d + \tfrac{\lambda_{max}(\mathbf{\Lambda})}{2}\left[1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q(1+\|\mathbf{q}\|)\right]\right] \\
&\quad +\dot{\bar{v}}_\omega + 2\left[\dot{\bar{v}}_q(2\bar{v}_q+1)+\bar{v}_q\dot{\bar{v}}_{q4}\right]\bar{w}_d + 2\bar{v}_q(\bar{v}_q+1)\dot{\bar{w}}_d \\
&\quad +\lambda_{max}(\mathbf{\Lambda})\left[\dot{\bar{v}}_{q4}\|\mathbf{q}\|+\dot{\bar{v}}_q(1+\|\mathbf{q}\|)\right].
\end{aligned}
$$

It can readily be seen that $b(\|\mathbf{q}\|)$, $c(\|\mathbf{q}\|)$, $d(\|\mathbf{q}\|)$, $e(\|\mathbf{q}\|)$ and $f(\|\mathbf{q}\|)$ are all strictly increasing functions of $\|\mathbf{q}\|$. Since no prior assumption can be made about $\|\mathbf{q}\|$ in the selection of the deadzone size, we must choose the worst case condition, $\|\mathbf{q}\| \le 1$. Accordingly, the deadzone size satisfies

$$
\Omega_m > \frac{b(1)+d(1)+f(1)+\bar{\boldsymbol{\tau}}_d/\underline{\lambda}_I}{\lambda_{min}(\mathbf{K})-(a+c(1)+e(1))} \tag{60}
$$

We now present an iterative algorithm for finding an ultimate upper bound on $\|\mathbf{q}(t)\|$ and $\|\delta\boldsymbol{\omega}(t)\|$ for the closed-loop system (52).

Algorithm 2

Set $\bar{q}_0 = 1$. For $i = 1, ..., n$, where n is some (user determined) finite positive integer, compute

$$
\begin{aligned}
\bar{v}^i_{\tilde{\omega}_d} &= \bar{q}_{i-1}\lambda_{max}(\mathbf{\Lambda})\left[1-\sqrt{1-\bar{v}_q^2}+\bar{v}_q\right] + \left[2(\bar{v}_q+1)+\lambda_{max}(\mathbf{\Lambda})\right]\bar{v}_q, \\
\bar{v}^i_{\tilde{\omega}} &= \bar{v}_\omega + \bar{v}^i_{\tilde{\omega}_d}, \\
\bar{r}_i &= \Omega_m + \bar{v}^i_{\tilde{\omega}}, \\
\bar{q}_i &= \tfrac{\bar{r}_i}{\lambda_{min}(\mathbf{\Lambda})}.
\end{aligned} \tag{61}
$$

Theorem 2

Assume that $0 \leq \bar{q}_1 < 1$ and $\lambda_{min}(\mathbf{K}) > a + c(1) + e(1)$. Then, the sequences $\{\bar{r}_i\}$ and $\{\bar{q}_i\}$ in Algorithm 3 are decreasing and convergent. Furthermore, the control and adaptation laws (41), (42) and (43) applied to the attitude dynamics (3) give

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\leq \bar{q}_i, \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\leq \left(\frac{\lambda_{max}(\boldsymbol{\Lambda})}{\lambda_{min}(\boldsymbol{\Lambda})} + 1\right)\bar{r}_i, \end{aligned}$$

for $i \geq 1$.

Proof

The proof that the sequences $\bar{r}_i$ and $\bar{q}_i$ are decreasing and convergent is similar to that in the proof of Theorem 1.

Consider the Lyapunov-like function

$$V(\tilde{\boldsymbol{\omega}}^m, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}_p) = \frac{1}{2}\tilde{\boldsymbol{\omega}}^{mT}\tilde{\boldsymbol{\omega}}^m + \frac{1}{2\gamma}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} + \frac{1}{2\gamma_p}\tilde{\boldsymbol{\theta}}_p^T\tilde{\boldsymbol{\theta}}_p. \tag{62}$$

Define the set $H = \left\{\tilde{\boldsymbol{\omega}}^m \in R^3 : \|\tilde{\boldsymbol{\omega}}^m\| \leq \Omega_m\right\}$. Using Property 1 for the projection operators and (60),

$$\dot{V} \leq -\Omega_m\left(\lambda_{min}(\mathbf{K}) - (a + c(1) + e(1))\right)\left[\Omega_m - \frac{b(1) + d(1) + f(1) + \bar{\boldsymbol{\tau}}_d/\underline{\lambda}_I}{\lambda_{min}(\mathbf{K}) - (a + c(1) + e(1))}\right] < 0, \tag{63}$$

when $\tilde{\boldsymbol{\omega}}^m$ is outside H. Since $\dot{\tilde{\boldsymbol{\theta}}} = \mathbf{0}$ and $\dot{\tilde{\boldsymbol{\theta}}}_p = \mathbf{0}$ when $\tilde{\boldsymbol{\omega}} \in H$, we can use the exact same arguments as in [14] and [15] to conclude: 1) $V(t)$ is bounded, which implies that $\tilde{\boldsymbol{\omega}}^m$ is bounded ($\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}_p$ are already guaranteed to be bounded by Property 2 of the projection operator), 2) if for some $t_1 \geq 0$, $\tilde{\boldsymbol{\omega}}(t_1)$ is outside H, there must exist a finite $t_2 > t_1$ such that $\tilde{\boldsymbol{\omega}}^m(t_2) \in H$, and 3) any intervals for which $\tilde{\boldsymbol{\omega}}^m(t)$ is outside H must shrink to zero as $t \to \infty$.

Since $\boldsymbol{\omega}_d(t)$, it follows from (28) that $\bar{\boldsymbol{\omega}}_d$ is bounded also. From (18) and (20), we find that $\tilde{\boldsymbol{\omega}}$ is bounded, such that by (29), (11), (15) and the fact that $\dot{\boldsymbol{\omega}}_d(t)$ is bounded, $\dot{\bar{\boldsymbol{\omega}}}_d$, $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^m$ are bounded also. By (18) and (20) it follows that $\dot{\bar{\boldsymbol{\omega}}}_d^m$ is bounded. Consequently, from (41) we find that $\boldsymbol{\tau}_c$ is bounded, and finally from (55) and (59), $\dot{\tilde{\boldsymbol{\omega}}}^m$ is bounded. Therefore, $\|\dot{\tilde{\boldsymbol{\omega}}}\| \leq \sigma$ for some $\sigma > 0$. Let (t_1, t_2) be an open interval for which $\tilde{\boldsymbol{\omega}}(t)$ is outside H, with $\|\tilde{\boldsymbol{\omega}}(t_1)\| = \|\tilde{\boldsymbol{\omega}}(t_2)\| = \Omega$. Then, $\|\tilde{\boldsymbol{\omega}}(t)\| \leq \Omega + \sigma(t_2 - t_1)$ for all $t \in (t_1, t_2)$. Since the intervals shrink to zero, this implies that given any $\epsilon' > 0$, there exists a $T' \geq 0$ such that $\|\tilde{\boldsymbol{\omega}}^m(t)\| \leq \Omega_m + \epsilon'$ for all $t \geq T'$.

Now, suppose that for some i, given any $\epsilon_1 > 0$, there exists a $T_{i-1}(\epsilon_1) \geq T'$ such that for all $t \geq T_{i-1}(\epsilon_1)$, $\|\mathbf{q}(t)\| \leq \bar{q}_{i-1} + \epsilon_1$. This condition clearly holds for $i = 1$ with $T_{i-1}(\epsilon_1) = T'$. Then, since $\|\tilde{\boldsymbol{\omega}}\| \leq \|\tilde{\boldsymbol{\omega}}^m\| + \bar{v}_{\tilde{\omega}}(\|\mathbf{q}\|)$, it follows that $\|\tilde{\boldsymbol{\omega}}(t)\| \leq \Omega_m + \epsilon' + \bar{v}_{\tilde{\omega}}(\bar{q}_{i-1} + \epsilon_1) = \bar{r}_i + \epsilon' + \bar{\epsilon}_1$ (where $\bar{\epsilon}_1 = \epsilon_1\lambda_{max}(\boldsymbol{\Lambda})\left[1 - \sqrt{1 - \bar{v}_q^2} + \bar{v}_q\right]$), for all $t \geq T_{i-1}(\epsilon_1)$. This shows that

$$\limsup_{t\to\infty} \|\tilde{\boldsymbol{\omega}}(t)\| \leq \bar{r}_i.$$

From Lemma 1, given any $\epsilon_2 > 0$, there exists a $T_i(\epsilon_2) \geq T_{i-1}(\epsilon_1)$ such that $\|\mathbf{q}(t)\| \leq \frac{\bar{r}_i + \epsilon' + \bar{\epsilon}_1}{\lambda_{min}(\boldsymbol{\Lambda})} + \epsilon_2$. Now, given any

$\epsilon_3 > 0$, it is possible to find $\epsilon' > 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\frac{\bar{r}_i+\epsilon'+\bar{\epsilon}_1}{\lambda_{min}(\mathbf{\Lambda})} + \epsilon_2 = \bar{q}_i + \epsilon_3$. Therefore, by induction, we conclude that $\limsup_{t\to\infty} \|\mathbf{q}(t)\| \leq \bar{q}_i$ for all $i \geq 1$. Finally, since $\|\delta\boldsymbol{\omega}\| \leq \|\tilde{\boldsymbol{\omega}}\| + \lambda_{max}(\mathbf{\Lambda})\|\mathbf{q}\|$, the conclusion for $\|\delta\boldsymbol{\omega}(t)\|$ follows. This concludes the proof. $\square$

Corollary 3

Let the conditions of Theorem 4 hold, and let $\bar{q} = \lim_{i\to\infty} \bar{q}_i$ *and* $\bar{r} = \lim_{i\to\infty} \bar{r}_i$. *Then,*

$$\begin{aligned} \limsup_{t\to\infty} \|\mathbf{q}(t)\| &\leq \bar{q}, \\ \limsup_{t\to\infty} \|\delta\boldsymbol{\omega}(t)\| &\leq \left(\frac{\lambda_{max}(\mathbf{\Lambda})}{\lambda_{min}(\mathbf{\Lambda})} + 1\right)\bar{r}. \end{aligned}$$

6 Conclusions

A result on the filtered error for attitude control problems has been presented, demonstrating that if the filtered error is ultimately upper bounded, then so are the quaternion and body-rate with bounds proportional to the filtered error ultimate upper bound. Making use of this result, bounds on the steady-state attitude and body-rate errors are derived for the spacecraft attitude tracking problem in the presence of model uncertainties and measurement errors. Both non-adaptive and adaptive control have been addressed. The resulting bounds can provide useful tools for attitude control system designers to assist in gain selection when specifications on steady-state attitude errors need to be satisfied.

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