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Finite-time output feedback attitude control for rigid spacecraft under control input saturation

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Abstract

In this paper, the problem of finite-time output feedback attitude control for rigid spacecraft subject to control input saturation is investigated. For the attitude stabilization, we propose a nonlinear finite-time observer using quaternion-based attitude representation. The estimated attitude of the observer always satisfies the unit norm constraint. The finite-time stability of the observer can be guaranteed by using homogeneous and Lyapunov methods. Furthermore, the effect of the external disturbance on the performance of the finite-time observer is analyzed. With the finite-time observer, we design an attitude stabilizing control law to guarantee that the attitude state of the spacecraft converges to the equilibrium point in finite time. We also discuss the finite-time attitude tracking control for spacecraft and propose a control scheme without requiring angular velocity measurements to guarantee that the desired attitude can be tracked in finite time. The main novelty of the control algorithms derived here lies in the fact that finite-time stability can be achieved even in the absence of angular velocity measurements as well as in the presence of constraints on control input magnitude. Numerical examples are presented to demonstrate the efficiency of the proposed control algorithms.

1. Introduction

Attitude control of rigid spacecraft has been a benchmark control problem and has been extensively investigated for many years because of its wide applications in the area of spacecraft missions. Most of the attitude control laws in the existing available literature can only achieve asymptotic stability, which implies that the convergence of system trajectories to the equilibrium state over an infinite horizon. As compared with the asymptotic control law, finite-time control law can provide faster convergence rate, higher precision control performance and better disturbance rejection property for rigid spacecraft attitude control, which leads to an enhanced application efficiency of spacecraft. Therefore, the development of a finite-time attitude control scheme is an interesting and challenging problem.

There is a series of methods that can be used to design an attitude control law with finite-time convergence, such as the terminal sliding mode method [1, 2, 3, 4, 5], the technique of adding a power integrator [6], and the homogeneous method [7]. Using the standard terminal sliding mode control [8], two finite-time attitude tracking controllers were proposed for rigid spacecraft [1], and a robust attitude stabilization control law with finite-time convergence was developed in [2] for rigid spacecraft in the presence of external disturbance and inertia uncertainty. Unfortunately, the main disadvantage of the standard terminal sliding mode control is the singularity problem. Therefore, the nonsingular

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terminal sliding mode manifold [9] and the modified terminal sliding mode manifold [10] were proposed to eliminate the singularity problem associated with the standard terminal sliding mode control. In [3], the nonsingular terminal sliding mode developed in [9] was employed to design a finite-time attitude controller. Based on the modified terminal sliding mode manifold [10] and Chebyshev neural network, a finite-time attitude tracking control scheme was presented for spacecraft [5]. Using the fast nonsingular terminal sliding mode control, the finite-time attitude tracking control problem for rigid spacecraft with external disturbances and inertia uncertainties was addressed in [4]. By application of the technique of adding a power integrator [11], a finite-time attitude tracking scheme was proposed in [6]. The homogeneous method was employed in [7] to design a finite-time attitude stabilization control algorithm for rigid spacecraft subject to control input saturation.

It is worth noticing that full state measurements (i.e., both attitude and angular velocity) are required to implement the above-mentioned finite-time attitude controllers. In contrast, a finite-time attitude tracking control law without requiring angular velocity measurement was proposed for rigid spacecraft with the attitude being represented by the modified Rodrigues parameters (MRPs) [12]. The finite-time output feedback attitude stabilization problem has been studied in [13] based on the MRPs attitude representation. However, the MRP representation is not global because of the singularity, and the control input saturation has not been taken into account. In practical applications, a spacecraft suffers from the control input saturation due to the physical limits of the onboard actuators. The controller designer must address this issue as it may lead to instability or unacceptable performance degradation of the spacecraft's attitude.

In this paper, we focus on the finite-time output feedback attitude control problem for rigid spacecraft with the attitude being represented by the unit quaternion and subject to control input saturation. By using finite-time observer technique, a nonlinear finite-time observer is proposed for rigid spacecraft. The attitude state of the finite-time observer can always satisfy the unit norm constraint. Then, two novel finite-time attitude controllers are designed for rigid spacecraft based on finite-time observer technique and homogeneous method. In contrast to the existing results in [1, 2, 3, 4, 5, 6, 7], the requirement of angular velocity measurement is no longer necessary in the proposed control schemes. Different from the finite-time output feedback attitude controllers reported in [12, 13], the proposed control laws employ the unit quaternion as the attitude representation and always satisfy the constraint on the input magnitude. It should be emphasized that it is nontrivial to apply the method in [12, 13] to design a finite-time output feedback attitude controller for rigid spacecraft using quaternion-based attitude representation and subject to control input saturation.

This paper is organized as follows. Some basics for the spacecraft attitude dynamics and homogeneity are described in Section 2. Section 3 proposes a velocity-free finite-time attitude stabilizing control law for spacecraft subject to control input saturation, and Section 4 discusses the problem of finite-time output feedback attitude tracking control for rigid spacecraft. Simulation results are presented in Section 5 followed by conclusions in Section 6.

2. Background and Preliminaries

2.1. Notations

The notation $\|\cdot\|$ refers to the Euclidean norm of a vector or the induced norm of a matrix. I_n represents the $n\times n$ identity matrix. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues of a matrix, respectively. Given a vector $x\in R^n$ and $\alpha\geq 0$, define $\mathrm{sig}^{\alpha}(x)=[\mathrm{sig}^{\alpha}(x_1),\mathrm{sig}^{\alpha}(x_2),\cdots,\mathrm{sig}^{\alpha}(x_n)]^T$, $\mathrm{diag}\,(|x|^{\alpha})=\mathrm{diag}\,(|x_1|^{\alpha},|x_2|^{\alpha},\cdots,|x_n|^{\alpha})$, $\mathrm{sat}_{\alpha}(x)=[\mathrm{sat}_{\alpha}(x_1),\mathrm{sat}_{\alpha}(x_2),\cdots,\mathrm{sat}_{\alpha}(x_n)]^T$, and $x^{\alpha}=[x_1^{\alpha},x_2^{\alpha},\cdots,x_n^{\alpha}]^T$, where $\mathrm{sig}^{\alpha}(x_i)=\mathrm{sgn}(x_i)|x_i|^{\alpha}(i=1,2,\cdots,n)$, $\mathrm{sgn}(\cdot)$ denotes the signum function, and $\mathrm{sat}_{\alpha}(x_i)$ is defined by [7]

$$\mathrm{sat}_{\alpha}(x_i)=\begin{cases}\mathrm{sgn}(x_i), & \text{if} \quad |x_i|>1\\ \mathrm{sig}^{\alpha}(x_i), & \text{if} \quad |x_i|\leq 1.\end{cases} \tag{1}$$

Note that $\alpha=1$, $\mathrm{sat}_{\alpha}(\cdot)$ becomes the standard saturation function $\mathrm{sat}(\cdot)=\mathrm{sat}_1(\cdot)$.

2.2. Spacecraft Attitude Dynamics and Kinematics

The spacecraft is modeled as a rigid body with actuators that provide torques about three mutually perpendicular axes that define a body-fixed frame B. The equations of motion are given by [14]

$$\dot{q} = \frac{1}{2}\begin{bmatrix} q_4 I_3 + \bar{q}^{\times} \\ -\bar{q}^T \end{bmatrix}\omega = \frac{1}{2}A(q)\omega \tag{2}$$

$$J\dot{\omega} = -\omega^{\times} J\omega + \tau \tag{3}$$

where $\omega = [\omega_1, \omega_2, \omega_3]^T$ is the angular velocity of the spacecraft with respect to an inertial frame I and expressed in the body frame B, $J \in R^{3\times3}$ is the positive-definite mass moment of inertia matrix, $\tau \in R^3$ is the applied control torque generated by actuators, and $x^{\times} \in R^{3\times3}$ for a vector $x = [x_1, x_2, x_3]^T$ denotes the skew-symmetric matrix given by

$$x^{\times} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{4}$$

The limit on the control torque is considered in this paper as $|\tau_i| \leq \tau_M (i = 1, 2, 3)$, where τ_M is a positive constant.

The unit quaternion $q = [\bar{q}^T, q_4]^T = [q_1, q_2, q_3, q_4]^T$ represents the attitude of a rigid spacecraft in the body frame B with respect to the inertial frame I, which is defined by

$$\bar{q} = [q_1, q_2, q_3]^T = \kappa \sin(\kappa_0/2), \quad q_4 = \cos(\kappa_0/2) \tag{5}$$

where κ is the Euler axis, and κ_0 is the Euler angle. The unit quaternion q satisfies the constraint $\|q\| = 1$. Let q represent a given attitude, then $-q$ represents the same attitude after a rotation of $\pm 2\pi$ about an arbitrary axis. The inverse of a quaternion q is given by $q^{-1} = [-\bar{q}^T, q_4]^T$. The quaternion multiplication [15] of two unit quaternion q and q_o is denoted by $q \odot q_o$.

Two control objectives of the present work are: (i) design a control torque τ such that the attitude state of the spacecraft can converge to the equilibrium point in finite time and (ii) determine a control law for τ such that the attitude state of the spacecraft can track a time-varying reference trajectory in finite time. The above control objectives should be achieved even in the presence of control input saturation as well as in the absence of angular velocity measurement.

2.3. Homogeneity

For any $\lambda > 0$ and any set of real parameters $r_i > 0$ $(i = 1, 2, \cdots, n)$, a dilation operator $\delta_\lambda^r : R^n \longmapsto R^n$ is defined by

$$\delta_\lambda^r(x_1, x_2, \cdots, x_n) = (\lambda^{r_1} x_1, \lambda^{r_2} x_2, \cdots, \lambda^{r_n} x_n) \tag{6}$$

where $r = [r_1, r_2, \cdots, r_n]^T$.

A continuous function $f : R^n \longmapsto R$ is homogeneous of degree k with respect to the dilation δ_λ^r if

$$\forall \lambda > 0,\ f(\delta_\lambda^r(x)) = \lambda^k f(x). \tag{7}$$

A differential system $\dot{x} = f(x)$ (or a vector field f), with continuous $f : R^n \longmapsto R^n$, is homogeneous of degree k with respect to the dilation δ_λ^r if

$$\forall \lambda > 0,\ f_i(\delta_\lambda^r(x)) = \lambda^{k+r_i} f_i(x), i = 1, 2, \cdots, n. \tag{8}$$

2.4. Definitions and Lemmas

Definition 1 [16]. Consider the following system:

$$\dot{x} = f(x, t),\ f(0, t) = 0, x \in U \subset R^n \tag{9}$$

where $f : U \times R^+ \to R^n$ is continuous on an open neighborhood U of the origin $x = 0$. The zero solution of (9) is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U_0 \subseteq U$ of the origin. The "finite-time convergence" means: If, for any initial condition $x(t_0) = x_0 \in U_0$ at any given initial time t_0, there is a setting time $T > 0$, such that every every solution $x(t; t_0, x_0)$ of system (9) is defined with $x(t; t_0, x_0) \in U_0 \setminus \{0\}$ for $t \in [t_0, T)$ and

$$\lim_{t \to T} x(t; t_0, x_0) = 0; \quad x(t; t_0, x_0) = 0, \quad \forall t > T. \tag{10}$$

When $U = U_0 = R^n$, the zero solution is said to be globally finite-time stable.

Lemma 1 [16, 17]. Suppose that there is a Lyapunov function $V(x, t)$ defined on $U_1 \times R^+$, where $U_1 \subseteq U \in R^n$ is a neighborhood of the origin, and

$$\dot{V}(x, t) \leq -lV^a(x, t), \forall x \in U_1 \setminus \{0\} \tag{11}$$

where $l > 0$ and $0 < a < 1$. Then, the origin of system (9) is locally finite-time stable. The settling time satisfies

$$T \leq \frac{V^{1-a}(x(t_0), t_0)}{l(1 - a)} \tag{12}$$

for a given initial condition $x(t_0) \in U_1$.

Corollary 1. Suppose that there is a Lyapunov function $V(x, t)$ defined on $U \times R^+$, and

$$\dot{V}(x, t) \leq -lV^a(x, t) + kV(x, t), \forall x \in U \setminus \{0\} \tag{13}$$

where $l, k > 0$ and $0 < a < 1$. Then, for a given initial condition $x(t_0)$ at any initial time t_0, the origin of system (9) is locally finite-time stable if $x(t_0) \in U_1 \subseteq U$, where $U_1 = \{x | V^{1-a}(x, t) < l/k\}$ is a neighborhood of the origin. The settling time satisfies

$$T \leq \frac{V^{1-a}(x(t_0), t_0)}{(l - kV^{1-\alpha}(x(t_0), t_0))(1 - a)} \tag{14}$$

for a given initial condition $x(t_0) \in U_1$.

Proof. See the Appendix.

Lemma 2 [11]. For any $x \in R, y \in R, c > 0$, and $d > 0$, $|x|^c |y|^d \leq c|x|^{c+d}/(c + d) + d|y|^{c+d}/(c + d)$.

Lemma 3 [6]. For any $x_i \in R, i = 1, 2, \cdots, n$, and a real number $\nu \in (0, 1]$, $(\sum_{i=1}^n |x_i|)^\nu \leq \sum_{i=1}^n |x_i|^\nu \leq n^{1-\nu} \left(\sum_{i=1}^n |x_i|\right)^\nu$.

Proof. See the Appendix.

Lemma 4 [18]. For any $x_i \in R, i = 1, 2, \cdots, n$, and a real number $p > 1$, $\sum_{i=1}^n |x_i|^p \leq \left(\sum_{i=1}^n |x_i|\right)^p \leq n^{p-1} \sum_{i=1}^n |x_i|^p$.

Lemma 5 [13]. Consider the following system:

$$\dot{x} = f(x) + \hat{f}(x, y), f(0) = 0, x \in R^n \tag{15}$$

where $f(x)$ is a continuous homogeneous vector field of degree $k < 0$ with respect to the dilation δ^r_λ, $\hat{f}(0, y) = 0$ and y is a bounded vector function. Assume that $x = 0$ is an asymptotically stable equilibrium of the system $\dot{x} = f(x)$. Then, $x = 0$ is a local finite-time stable equilibrium of system (15) if and only if

$$\lim_{\lambda \to 0} \frac{\hat{f}_i(\lambda^{r_1} x_1, \lambda^{r_2} x_2, \cdots, \lambda^{r_n} x_n, y)}{\lambda^{r_i + k}} = 0, i = 1, 2, \cdots, n, \forall x \neq 0. \tag{16}$$

Lemma 6 [19]. Let the matrix $J \in R^{3\times3}$ be symmetric and positive definite. Then, for any $x \in R^3$, the following inequality holds:

$$||x^\times Jx|| \leq (\lambda^2_{max}(J) - \lambda^2_{min}(J))^{1/2} ||x||^2. \tag{17}$$

3. Velocity-Free Finite-Time Attitude Stabilizing Control of Spacecraft

In this section, a velocity-free finite-time output feedback attitude controller is designed such that the attitude of the spacecraft can converge to the equilibrium point in finite time even in the presence of control input saturation. We assume that attitude measurements are available for feedback. An attitude measurement can be obtained by use of a star-tracker (provides attitude measurement directly), or by using a pair of non-collinear vector measurements (such as a sun-sensor and magnetometer) and applying an algorithm such as QUEST to obtain an attitude measurement. Before designing the controller, a finite-time observer is introduced to estimate the unmeasurable angular velocity in finite time.

3.1. Finite-Time Observer

The finite-time observer for the spacecraft described by (2) and (3) is designed as follows:

$$\dot{\hat{q}} = \frac{1}{2}A(\hat{q})C_{\tilde{q}}^{-1}\left[\hat{\omega} + \theta\gamma_1 P^{-1}\mathrm{sig}^{\alpha}(\bar{\tilde{q}}) + \frac{2\gamma_3\bar{\tilde{q}}}{\tilde{q}_4(1-\bar{\tilde{q}}^T\bar{\tilde{q}})}\right], \quad \hat{q}(0) = q(0) \tag{18}$$

$$J\dot{\hat{\omega}} = -\hat{\omega}^{\times}J\hat{\omega} + \tau + \theta^2\gamma_2 J\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}), \quad \hat{\omega}(0) = 0 \tag{19}$$

where $\hat{q}$ and $\hat{\omega}$ are estimates of q and ω, respectively, $\alpha \in (1/2, 1)$, $\alpha_1 = 2\alpha - 1$, $\theta, \gamma_1, \gamma_2$ and γ_3 are some positive constants, respectively, $\tilde{q} = \hat{q}^{-1} \odot q$ is the quaternion error between q and $\hat{q}$, $\bar{\tilde{q}} = [\tilde{q}_1, \tilde{q}_2, \tilde{q}_3]^T$, $P = (\tilde{q}_4 I_3 + \bar{\tilde{q}}^{\times})/2$, and $C_{\tilde{q}}$ denotes the corresponding direction cosine matrix relate to the quaternion $\tilde{q}$ given by

$$C_{\tilde{q}} = C(\tilde{q}) = \left(\tilde{q}_4^2 - \bar{\tilde{q}}^T\bar{\tilde{q}}\right)I_3 + 2\bar{\tilde{q}}\bar{\tilde{q}}^T - 2\tilde{q}_4\bar{\tilde{q}}^{\times}. \tag{20}$$

Note that $\dot{C}_{\tilde{q}} = -\left[\tilde{\omega} - \theta\gamma_1 P^{-1}\mathrm{sig}^{\alpha}(\bar{\tilde{q}}) - 2\gamma_3\bar{\tilde{q}}/(\tilde{q}_4(1-\bar{\tilde{q}}^T\bar{\tilde{q}}))\right]^{\times} C_{\tilde{q}}$, where $\tilde{\omega} = \omega - \hat{\omega}$ is the angular velocity estimation error. Then, the dynamic equations for observer errors $\tilde{q}$ and $\tilde{\omega}$ are obtained from (2), (3), (18) and (19) as follows:

$$\dot{\tilde{q}} = \frac{1}{2}A(\tilde{q})\left[\tilde{\omega} - \theta\gamma_1 P^{-1}\mathrm{sig}^{\alpha}(\bar{\tilde{q}}) - \frac{2\gamma_3\bar{\tilde{q}}}{\tilde{q}_4(1-\bar{\tilde{q}}^T\bar{\tilde{q}})}\right] \tag{21}$$

$$\dot{\tilde{\omega}} = -J^{-1}(\omega^{\times}J\omega - \hat{\omega}^{\times}J\hat{\omega}) - \theta^2\gamma_2\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}). \tag{22}$$

For the stability analysis, the governing equations for observer errors $\bar{\tilde{q}}$ and $\tilde{\omega}$ are rewritten as follows:

$$\begin{cases} \dot{\bar{\tilde{q}}} = \frac{1}{2}\tilde{\omega} - \theta\gamma_1\mathrm{sig}^{\alpha}(\bar{\tilde{q}}) + f_1(\tilde{q},\ \tilde{\omega}) \\ \dot{\tilde{\omega}} = -\theta^2\gamma_2\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}) + f_2(\omega,\ \hat{\omega}) \end{cases} \tag{23}$$

where $f_1(\tilde{q},\ \tilde{\omega}) = (P - I_3/2)\tilde{\omega} - 2\gamma_3 P\bar{\tilde{q}}/\left[\tilde{q}_4(1-\bar{\tilde{q}}^T\bar{\tilde{q}})\right]$ and $f_2(\omega,\ \hat{\omega}) = -J^{-1}(\omega^{\times}J\omega - \hat{\omega}^{\times}J\hat{\omega})$.

Corollary 2. For any given constant $\Delta > 0$, if $\hat{q}$ is generated by (18), $\|\omega\| \leq \Delta$, and $\|\hat{\omega}\| \leq \Delta$, then $\|\bar{\tilde{q}}(t)\| \leq \Delta_1, \forall t \in [0, \infty)$, where $\Delta_1 = \sqrt{\Delta^2/(\Delta^2 + \theta\gamma_1\gamma_3)} < 1$.

Proof. See the Appendix.

Remark 1. Note that $\hat{q}(0) = q(0)$, which implies that $\|\hat{q}(0)\| = \|q(0)\| = 1$. Consider $V = \hat{q}^T\hat{q}$. Then, the time derivative of V along with system (18) is $\dot{V} \equiv 0$, where the fact that $\hat{q}^T A(\hat{q}) = 0$ has been used. Therefore, $\|\hat{q}(t)\| = 1$ for any $t \geq 0$. Furthermore, since q and $\hat{q}$ satisfy the unit norm constraint, $\tilde{q}$ also satisfies the unit norm constraint.

Remark 2. From Corollary 2, we know that if ω and $\hat{\omega}$ are bounded for all time, then $\|\bar{\tilde{q}}(t)\| \leq \Delta_1 < 1, \forall t \in [0, \infty)$, which implies that $\tilde{q}_4(t) > 0, \forall t \in [0, \infty)$. Therefore, the possible singularity problem in (18) will never occur if the designed control law can guarantee that ω and $\hat{\omega}$ are bounded for all time.

Theorem 1. Consider the observer defined by (18) and (19). For any positive constant Δ, if $\|\omega\| \leq \Delta$, $\|\hat{\omega}\| \leq \Delta$, and the parameter θ is chosen to be sufficiently large, then the zero solution of system (23) is locally finite-time stable.

Proof. To facilitate the stability analysis, we introduce the following coordinate transformation: $\zeta_1 = \bar{\tilde{q}}/\theta$ and $\zeta_2 = \tilde{\omega}/\theta^2$. Then, the error dynamics in terms of the transformed coordinate is given by

$$\begin{cases} \dot{\zeta}_1 = \frac{\theta}{2}\zeta_2 - \theta^{\alpha}\gamma_1 \mathrm{sig}^{\alpha}(\zeta_1) + \frac{f_1(\tilde{q},\,\tilde{\omega})}{\theta} \\ \dot{\zeta}_2 = -\theta^{\alpha_1}\gamma_2 \mathrm{sig}^{\alpha_1}(\zeta_1) + \frac{f_2(\omega,\,\hat{\omega})}{\theta^2}. \end{cases} \tag{24}$$

Let $M \in R^{6\times 6}$ be

$$M = \begin{bmatrix} -\gamma_1 I_3 & \frac{1}{2} I_3 \\ -\gamma_2 I_3 & 0 \end{bmatrix}. \tag{25}$$

Because M is a Hurwitz matrix, there exists a positive definite matrix $N = N^T$ such that $M^T N + NM = -I_6$.

Define $\zeta = [\zeta_1^T, \zeta_2^T]^T$, $\chi_1 = \zeta_1$, $\chi_2 = \mathrm{sig}^{1/\alpha}(\zeta_2)$, $\chi = [\chi_1^T, \chi_2^T]^T$, the dilation $\delta_\lambda^r(\zeta) = (\lambda\zeta_1^T, \lambda^{\alpha}\zeta_2^T)$, and

$$V_\alpha(\zeta) = \chi^T N \chi. \tag{26}$$

First, we consider the case when $f_1(\tilde{q},\tilde{\omega}) = 0$ and $f_2(\omega,\ \hat{\omega}) = 0$, i.e., system (24) is

$$\begin{cases} \dot{\zeta}_1 = \frac{\theta}{2}\zeta_2 - \theta^{\alpha}\gamma_1 \mathrm{sig}^{\alpha}(\zeta_1) \\ \dot{\zeta}_2 = -\theta^{\alpha_1}\gamma_2 \mathrm{sig}^{\alpha_1}(\zeta_1). \end{cases} \tag{27}$$

Let f_α be the vector field of (27), and it can be verified that f_α is homogeneous of degree $\alpha - 1$ with respect to the dilation $\delta_\lambda^r(\zeta)$. Note that $-\alpha < \alpha - 1 < 0$ if $\alpha \in (1/2, 1)$. Furthermore, $V_\alpha(\zeta)$ and $L_{f_\alpha} V_\alpha(\zeta)$ are homogeneous of degree 2 and $\alpha + 1$ with respect to the dilation $\delta_\lambda^r(\zeta)$, respectively. By application of Lemma 4.2 in [20], we obtain that

$$-c_1(\alpha,\theta)V_\alpha^{\beta}(\zeta) \le L_{f_\alpha} V_\alpha(\zeta) \le -c_2(\alpha,\theta)V_\alpha^{\beta}(\zeta) \tag{28}$$

where $c_1(\alpha,\theta) = -\min_{\{z:V_\alpha(z)=1\}} L_{f_\alpha} V_\alpha(z)$, $c_2(\alpha,\theta) = -\max_{\{z:V_\alpha(z)=1\}} L_{f_\alpha} V_\alpha(z)$, and $\beta = (1+\alpha)/2$. Using the same method as that in [21] and [22], it can be verified that $\lim_{\alpha\to 1} c_2(\alpha,\theta) \ge \theta/\lambda_{\max}(N)$.

Next, the time derivative of V_α along with system (24) is obtained as follows

$$\dot{V}_\alpha(\zeta) \le -c_2(\alpha,\theta)V_\alpha^{\beta}(\zeta) + 2\chi^T N \begin{bmatrix} \frac{1}{\theta} f_1(\tilde{q},\ \tilde{\omega}) \\ \frac{1}{\alpha\theta^2}\mathrm{diag}\left(|\zeta_2|^{\frac{1}{\alpha}-1}\right) f_2(\omega,\ \hat{\omega}) \end{bmatrix}. \tag{29}$$

If we choose the observer parameter θ such that $\theta \ge \Delta^2$, then we have $\|\bar{\tilde{q}}(t)\| \le \Delta_1 \le \Delta_2 = \sqrt{1/(1+\gamma_1\gamma_3)} < 1$ and $\tilde{q}_4(t) \ge \sqrt{1-\Delta_2^2}$, $\forall t \in [0,\infty)$, where Corollary 2 has been used.

Noting that

$$\left\| \left(P - \frac{1}{2}I_3\right)\tilde{\omega} \right\| \le \|\tilde{\omega}\| \frac{|\tilde{q}_4 - 1| + \|\bar{\tilde{q}}\|}{2} \le 2\Delta\|\bar{\tilde{q}}\| \tag{30}$$

$$\left\| -\frac{2\gamma_3 P\bar{\tilde{q}}}{\tilde{q}_4(1-\tilde{q}^T\tilde{q})} \right\| \le \frac{\sqrt{2}\gamma_3}{(1-\Delta_2^2)^{3/2}}\|\bar{\tilde{q}}\| \tag{31}$$

$$\|J^{-1}(\omega^{\times}J\omega - \hat{\omega}^{\times}J\hat{\omega})\| \le 4\Delta\lambda_{\max}(J^{-1})\lambda_{\max}(J)\|\tilde{\omega}\| \tag{32}$$

where the inequality $\|P\| \le \sqrt{2}/2$ is applied, we obtain

$$2\chi^T N \begin{bmatrix} \frac{1}{\theta} f_1(\tilde{q},\ \tilde{\omega}) \\ \frac{1}{\alpha\theta^2}\mathrm{diag}\left(|\zeta_2|^{\frac{1}{\alpha}-1}\right) f_2(\omega,\ \hat{\omega}) \end{bmatrix} \le 2\lambda_{\max}(N)\|\chi\| \left[\rho_1\|\chi_1\| + \frac{\rho_2}{\alpha}\left(\sum_{i=1}^{3}|\zeta_{2i}|^{\frac{1}{\alpha}-1}\right)\left(\sum_{i=1}^{3}|\zeta_{2i}|\right)\right] \tag{33}$$

where $\rho_1 = 2\Delta + \sqrt{2}\gamma_3/(1-\Delta_2^2)^{3/2}$ and $\rho_2 = 4\Delta\lambda_{\max}(J^{-1})\lambda_{\max}(J)$.

By using Lemma 2, we obtain

$$|\zeta_{2j}|^{\frac{1}{\alpha}-1}\left(\sum_{i=1}^{3}|\zeta_{2i}|\right) \le 3(1-\alpha)|\zeta_{2j}|^{\frac{1}{\alpha}} + \alpha\left(\sum_{i=1}^{3}|\zeta_{2i}|^{\frac{1}{\alpha}}\right) \tag{34}$$

where $j = 1, 2, 3$, and it follows that

$$\left(\sum_{i=1}^{3} |\zeta_{2i}|^{\frac{1}{\alpha}-1}\right)\left(\sum_{i=1}^{3} |\zeta_{2i}|\right) \leq 3\left(\sum_{i=1}^{3} |\zeta_{2i}|^{\frac{1}{\alpha}}\right) \leq 3\sqrt{3}\|\chi_2\| \tag{35}$$

where Lemma 3 is used. Applying (35) to (33) leads to

$$2\chi^T N \begin{bmatrix} \frac{1}{\theta} f_1(\tilde{q},\ \tilde{\omega}) \\ \frac{1}{\alpha\theta^2}\mathrm{diag}\left(|\zeta_2|^{\frac{1}{\alpha}-1}\right) f_2(\omega,\ \hat{\omega}) \end{bmatrix} \leq 2\sqrt{2}\rho\lambda_{\max}(N)\|\chi\|^2 \leq c_3 V_\alpha(\zeta) \tag{36}$$

where $\rho = \max\{\rho_1, 3\sqrt{3}\rho_2/\alpha\}$, $c_3 = 2\sqrt{2}\rho\lambda_{\max}(N)/\lambda_{\min}(N)$, and the inequality $\|\chi_1\| + \|\chi_2\| \leq \sqrt{2}\|\chi\|$ is applied. Thus, (29) becomes

$$\dot{V}_\alpha(\zeta) \leq -c_2(\alpha,\theta)V_\alpha^\beta(\zeta) + c_3 V_\alpha(\zeta) \tag{37}$$

which implies that $\dot{V}_\alpha \leq 0$ as long as ζ lies within the following set:

$$\Omega_1 = \left\{\zeta \left| V_\alpha(\zeta) < \left(\frac{c_2}{c_3}\right)^{1/(1-\beta)}\right.\right\}. \tag{38}$$

By noting that $V_\alpha \leq \lambda_{\max}(N)\|\chi\|^2$, the set

$$\Omega_{10} = \left\{\zeta \left| \|\chi\| < \sqrt{\frac{1}{\lambda_{\max}(N)}\left(\frac{c_2}{c_3}\right)^{1/(1-\beta)}}\right.\right\}$$

is a subset of Ω_1.

Note that if $\|\bar{\tilde{q}}\| \leq \Delta_2$, $\|\omega\| \leq \Delta$, $\|\hat{\omega}\| \leq \Delta$, and $\theta > \max\{1, \Delta^2\}$, then $\|\zeta\| \leq \Delta_2/\theta + 2\Delta/\theta^2 \leq \Delta_3/\theta$, where $\Delta_3 = \Delta_2 + 2\Delta$, and χ can be bounded by

$$\|\chi\| \leq \|\zeta_1\| + \left\|\mathrm{sig}^{1/\alpha}(\zeta_2)\right\| \leq \|\zeta_1\| + \|\zeta_2\|^{1/\alpha} \leq \frac{1}{\theta}\left(\Delta_3 + \Delta_3^{1/\alpha}\right) \tag{39}$$

where Lemma and the inequality $1/\theta^{1/\alpha} \leq 1/\theta$ are applied. Thus, ζ lies in the following set:

$$\Omega_2 = \left\{\zeta \left| \|\chi\| \leq \frac{1}{\theta}\left(\Delta_3 + \Delta_3^{1/\alpha}\right)\right.\right\} \tag{40}$$

By Corollary 1, to show the finite-time stability of the observer, it must be guaranteed that $\Omega_2 \subseteq \Omega_1$. Let the parameter $\theta > \max\{1, \Delta^2\}$ be sufficiently large such that

$$\theta\sqrt{\frac{1}{\lambda_{\max}(N)}\left(\frac{c_2}{c_3}\right)^{\frac{1}{1-\beta}}} > \Delta_3 + \Delta_3^{\frac{1}{\alpha}} \tag{41}$$

which implies that $\Omega_2 \subset \Omega_{10} \subset \Omega_1$. Hence, we conclude that the origin of system (24) is locally finite-time stable, and consequently, the zero solution of system (23) is locally finite-time stable.

Remark 3. In Theorem 1, we show that $\bar{\tilde{q}}$ converges to zero in finite time. By the unit norm constraint (i.e., $\|\tilde{q}\| = 1$) and Corollary 2, we can conclude that $\tilde{q}_4$ converges to +1 in finite time.

Remark 4. In [21] and [22], finite-time observers have been proposed for an uniformly observable and globally Lipschitzian nonlinear system. Using MRPs as attitude representation, semi-global finite-time observers have been presented for rigid spacecraft [12, 13]. However, the three-parameters (MRPs) attitude representation suffers from singularity problems [23], and thus unit quaternion is chosen to represent the attitude of the spacecraft in this paper. It is also important to point out that, due to the unit norm constraint (i.e., $\|q(t)\| = 1, \forall t \in [0, \infty)$), the methods in [12, 13, 21, 22] cannot be applied for attitude control of rigid spacecraft using quaternion-based attitude representation.

Remark 5. As stated in proof of Theorem 1, the parameter $\theta > \max\{1, \Delta^2\}$ needs to be sufficiently large as we consider the bound of ω and $\hat{\omega}$ can be any positive constant. In practice, since the bound of ω and $\hat{\omega}$ is usually not a large number, we can choose the parameter θ larger than 1 rather than being sufficiently large. This can be explained as follows.

Note that the governing equations for observer errors $\tilde{\bar{q}}$ and $\tilde{\omega}$ are

$$\begin{cases} \dot{\tilde{\bar{q}}} = \frac{1}{2}\tilde{\omega} - \theta\gamma_1 \mathrm{sig}^{\alpha}(\tilde{\bar{q}}) + f_1(\tilde{q},\ \tilde{\omega}) \\ \dot{\tilde{\omega}} = -\theta^2\gamma_2 \mathrm{sig}^{\alpha_1}(\tilde{\bar{q}}) + f_2(\omega,\ \hat{\omega}) \end{cases} \tag{42}$$

It is easy to verify that the nominal system of (42) (i.e., $f_1 = 0$ and $f_2 = 0$) is asymptotical stable. Consider the dilation $\delta_\lambda^r(\tilde{\bar{q}}, \tilde{\omega}) = (\lambda\tilde{\bar{q}}^T, \lambda^\alpha\tilde{\omega}^T)$, and we can verify that the nominal system of (42) is homogeneous of degree $\alpha - 1 < 0$ with respect to the dilation $\delta_\lambda^r(\tilde{\bar{q}}, \tilde{\omega})$. Furthermore, by noting that $\tilde{q}_4 = \sqrt{1 - \tilde{\bar{q}}^T\tilde{\bar{q}}}$ and $f_2(\omega,\ \hat{\omega}) = -J^{-1}(\omega^\times J\omega - \hat{\omega}^\times J\hat{\omega}) = -J^{-1}(\omega^\times J\tilde{\omega} + \tilde{\omega}^\times J\hat{\omega}) = \bar{f}_2(\tilde{\bar{q}}, \tilde{\omega})$, it can be verified that

$$\lim_{\lambda\to 0} \frac{f_1(\lambda\tilde{\bar{q}}, \lambda^\alpha\tilde{\omega})}{\lambda^\alpha} = 0, \quad \lim_{\lambda\to 0} \frac{\bar{f}_2(\lambda\tilde{\bar{q}}, \lambda^\alpha\tilde{\omega})}{\lambda^{2\alpha-1}} = 0 \tag{43}$$

if ω and $\hat{\omega}$ are bounded. Using Lemma 5, we obtain that system (42) is locally finite-time stable without requiring θ to be sufficiently large.

Next, the effect of external disturbances on the performance of the finite-time observer is discussed. We assume that there are bounded external disturbances acting on the spacecraft, i.e., the attitude dynamics is

$$J\dot{\omega} = -\omega^\times J\omega + \tau + d \tag{44}$$

where $d \in R^3$ denotes the external disturbance and is bounded such that $\|d\| \le d_M$ with d_M being a positive constant. The result is stated as follows.

Corollary 3. Consider the spacecraft described by (2) and (44) in the presence of disturbance and the observer defined by (18) and (19). For any positive constant Δ, if $\|\omega\| \le \Delta$, $\|\hat{\omega}\| \le \Delta$, the observer parameter α satisfies $\alpha \in (1/2, 1)$ and the parameter θ is chosen to be sufficiently large, then the observer error $e = [\tilde{\bar{q}}^T, \tilde{\omega}^T]^T$ converges to the region $\|e\| \le \Delta_e$ in finite time, where Δ_e is defined by

$$\Delta_e = \frac{6^{1-\alpha/2}}{\lambda_{\min}^{\alpha/2}(N)} \left(\frac{c_4}{(c_2 - c_3)\theta^{2(1-\alpha)/\alpha}}\right)^{\alpha/(2\alpha-1)} \tag{45}$$

with c_4 given by

$$c_4 = \frac{2 \times 3^{\frac{1+\alpha}{2}} d_M \lambda_{\max}(N)}{\alpha(\lambda_{\min}(N))^{\frac{2-\alpha}{2}}}. \tag{46}$$

Proof. See the Appendix.

3.2. Finite-Time Attitude Stabilizing Controller Design

With the application of the finite-time observer, a velocity-free finite-time attitude controller is designed for rigid spacecraft as follows:

$$\tau = \begin{cases} -k_1 M_1^T \mathrm{sig}^{\alpha_1}(\bar{q}) - k_2 \mathrm{sat}_{\alpha_2}(\hat{\omega}), & \text{if} \quad (\bar{q}, \hat{\omega}) \in \Omega_3 \\ -k_1\bar{q} - k_2 \mathrm{sat}(\hat{\omega}), & \text{otherwise} \end{cases} \tag{47}$$

where k_1 and k_2 are positive constants, $M_1 = (q_4 I_3 + \bar{q}^\times)/2$, $\alpha_2 = \alpha_1/\alpha$, and Ω_3 is defined by

$$\Omega_3 = \left\{(\bar{q}, \hat{\omega}) \left| \sum_{i=1}^{3} |q_i|^{1+\alpha_1} + \frac{1+\alpha_1}{2k_1}\hat{\omega}^T J\hat{\omega} < 1 \right.\right\}. \tag{48}$$

It is easy to verify that M_1 is nonsingular if $(\bar{q}, \hat{\omega}) \in \Omega_3$.

It can be seen that the angular velocity measurement is not required in the control law (47). Moreover, the finite-time convergence of the attitude state can be achieved as shown in the following theorem. Hence, we call (47) as a velocity-free finite-time attitude controller. Next, we show that the finite-time controller (47) can satisfy the input saturation constraint if we choose some appropriate control gains k_1 and k_2.

If $(\bar{q}, \hat{\omega}) \in R^6 \setminus \Omega_3$, then $\tau_i (i = 1, 2, 3)$ can be bounded by $|\tau_i| \leq k_1 + k_2$. If $(\bar{q}, \hat{\omega}) \in \Omega_3$, then $\tau_1 = -k_1[q_4 \mathrm{sig}^{\alpha_1}(q_1) + q_3 \mathrm{sig}^{\alpha_1}(q_2) - q_2 \mathrm{sig}^{\alpha_1}(q_3)]/2 - k_2 \mathrm{sat}_{\alpha_2}(\hat{\omega}_1)$, and the bound on τ_1 is

$$\begin{aligned} |\tau_1| &\leq k_2 + \frac{k_1}{2} \left(|q_4||q_1|^{\alpha_1} + |q_3||q_2|^{\alpha_1} + |q_2||q_3|^{\alpha_1} \right) \\ &\leq k_2 + \frac{k_1 \left(|q_4|^{1+\alpha_1} + |q_3|^{1+\alpha_1} + |q_2|^{1+\alpha_1} \right)}{2(1+\alpha_1)} + \frac{k_1 \alpha_1 \left(|q_1|^{1+\alpha_1} + |q_2|^{1+\alpha_1} + |q_3|^{1+\alpha_1} \right)}{2(1+\alpha_1)} \\ &\leq k_2 + \frac{3^{1-\alpha} k_1}{2} \leq k_1 + k_2 \end{aligned} \tag{49}$$

where Lemmas 2 and 3 are used. Similarly, it can be verified that $|\tau_i| \leq k_1 + k_2 (i = 2, 3)$. Thus, the control input τ given in (47) can be bounded by $|\tau_i| \leq \tau_M (i = 1, 2, 3)$ if control gains k_1 and k_2 satisfy $k_1 + k_2 \leq \tau_M$.

Lemma 7. Consider the spacecraft described by (2) and (3) under the control law given by

$$\tau = -k_1 \bar{q} - k_2 \mathrm{sat}(\hat{\omega}) \tag{50}$$

where $\hat{\omega}$ is generated by the observer defined by (18) and (19). For any positive constant V_M, if the initial conditions satisfy $V_{10}(q_4(0), \hat{\omega}(0), \zeta(0)) \leq V_M$, where V_{10} is defined by

$$V_{10} = 2k_1(1 - q_4) + \frac{1}{2} \hat{\omega}^T J \hat{\omega} + V_\alpha(\zeta) \tag{51}$$

and $V_\alpha(\zeta)$ is given by (26), then there exists a sufficiently large observer parameter θ such that $\bar{q}$ and ω converge to zero as $t \to \infty$.

Proof. See the Appendix.

Theorem 2. Consider the spacecraft described by (2) and (3) under the control law (47), where $\hat{\omega}$ is generated by the observer defined by (18) and (19). For any positive constant V_M, if the initial conditions satisfy $V_{10}(q_4(0), \hat{\omega}(0), \zeta(0)) \leq V_M$, where V_{10} is defined by (51), then there exists a sufficiently large observer parameter θ such that $\bar{q}$ and ω converge to zero in finite time.

Proof. Two cases are considered for the proof, i.e., $(\bar{q}, \hat{\omega}) \in \Omega_3$ and $(\bar{q}, \hat{\omega}) \in R^6 \setminus \Omega_3$. When $(\bar{q}, \hat{\omega}) \in R^6 \setminus \Omega_3$, from Lemma 7, we conclude that $\bar{q}, \hat{\omega}$ and ω asymptotically converge to zero, which implies that $\bar{q}$ and $\hat{\omega}$ converge to the set Ω_3 in finite time.

When $(\bar{q}, \hat{\omega}) \in \Omega_3$, with the control law (47), the dynamic equations for $\bar{q}$ and $\hat{\omega}$ are described by

$$\dot{\bar{q}} = M_1 \hat{\omega} + M_1 \tilde{\omega} = M_1 \hat{\omega} + \theta^2 M_1 \zeta_2 \tag{52}$$

$$\begin{aligned} J \dot{\hat{\omega}} &= -\hat{\omega}^\times J \hat{\omega} + \theta^2 \gamma_2 J \mathrm{sig}^{\alpha_1}(\tilde{\bar{q}}) - k_1 M_1^T \mathrm{sig}^{\alpha_1}(\bar{q}) - k_2 \mathrm{sat}_{\alpha_2}(\hat{\omega}) \\ &= -\hat{\omega}^\times J \hat{\omega} - k_1 M_1^T \mathrm{sig}^{\alpha_1}(\bar{q}) - k_2 \mathrm{sat}_{\alpha_2}(\hat{\omega}) + \theta^{2+\alpha_1} \gamma_2 J \mathrm{sig}^{\alpha_1}(\zeta_1). \end{aligned} \tag{53}$$

Next, we show that all signals in the closed-loop system are bounded.

Consider the following Lyapunov function:

$$V_2 = \frac{k_1}{1+\alpha_1} \sum_{i=1}^{3} |q_i|^{1+\alpha_1} + \frac{1}{2} \hat{\omega}^T J \hat{\omega} + K_2 V_\alpha(\zeta) \tag{54}$$

where $V_\alpha(\zeta)$ is defined by (26), and $K_2 > 0$ is a sufficiently large constant.

The time derivative of V_2 along with (37), (52) and (53) is given by

$$\begin{aligned}
\dot{V}_2 &\leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) + k_1\theta^2 \left[\mathrm{sig}^{\alpha_1}(\bar{q})\right]^T M_1\zeta_2 + \theta^{2+\alpha_1}\gamma_2\hat{\omega}^T J\mathrm{sig}^{\alpha_1}(\zeta_1) - c_2K_2V_\alpha^\beta(\zeta) + c_3K_2V_\alpha(\zeta) \\
&\leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - c_2K_2V_\alpha^\beta(\zeta) + c_3K_2V_\alpha(\zeta) + \frac{\sqrt{2}\times 3^{(1-\alpha_1)/2}}{2}k_1\theta^2\|\zeta_2\| + 3^{(1-\alpha_1)/2}\gamma_2\theta^{2+\alpha_1}\lambda_{\max}(J)\|\hat{\omega}\|\|\zeta_1\|^{\alpha_1} \\
&\leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - c_2K_2V_\alpha^\beta(\zeta) + c_3K_2V_\alpha(\zeta) + \frac{\sqrt{2}\times 3^{(3-3\alpha)/2}}{2}k_1\theta^2\|\chi_2\|^\alpha + 3^{(1-\alpha_1)/2}\gamma_2\theta^{2+\alpha_1}\lambda_{\max}(J)\|\hat{\omega}\|\|\zeta_1\|^{\alpha_1} \\
&\leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - c_2K_2V_\alpha^\beta(\zeta) + c_3K_2V_\alpha(\zeta) + \rho_5V_\alpha^{\alpha/2}(\zeta) + \rho_6V_\alpha^{\alpha_1/2}(\zeta)
\end{aligned} \tag{55}$$

where the inequalities $\|\hat{\omega}\| \leq \sqrt{2V_M/\lambda_{\min}(J)}$, $\|M_1\| \leq \sqrt{2}/2$, $\|\mathrm{sig}^{\alpha_1}(\bar{q})\| \leq 3^{(1-\alpha_1)/2}$, $\|\mathrm{sig}^{\alpha_1}(\zeta_1)\| \leq 3^{(1-\alpha_1)/2}\|\zeta_1\|^{\alpha_1}$ and $\|\zeta_2\| \leq 3^{(1-\alpha)/2}\|\chi_2\|^\alpha$ are used, $\rho_5 = \sqrt{2}\times 3^{(3-3\alpha)/2}k_1\theta^2/[2\lambda_{\min}^{\alpha/2}(N)]$, and ρ_6 is defined by

$$\rho_6 = \frac{3^{(1-\alpha_1)/2}\sqrt{2V_M}\gamma_2\theta^{2+\alpha_1}\lambda_{\max}(J)}{\sqrt{\lambda_{\min}(J)\lambda_{\min}^{\alpha_1}(N)}} \tag{56}$$

Let $K_2 > \max\{\rho_5, \rho_6\}$, then (55) becomes

$$\dot{V}_2 \leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - c_2K_2V_\alpha^\beta(\zeta) + c_3K_2V_\alpha(\zeta) + K_2V_\alpha^{\alpha/2}(\zeta) + K_2V_\alpha^{\alpha_1/2}(\zeta). \tag{57}$$

Next, we consider two cases, i.e., $V_\alpha(\zeta) \geq 1$ and $V_\alpha(\zeta) < 1$. If $V_\alpha(\zeta) \geq 1$, then (57) becomes

$$\begin{aligned}
\dot{V}_2 &\leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - c_2K_2V_\alpha^\beta(\zeta) + (c_3+2)K_2V_\alpha(\zeta) \\
&= -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - K_2V_\alpha^\beta(\zeta)\left[c_2 - (c_3+2)V_\alpha^{1-\beta}(\zeta)\right].
\end{aligned} \tag{58}$$

We can choose a sufficiently large parameter θ such that $c_2 > (c_3+2)V_{M0}^{1-\beta}$, where $V_{M0} = V_M + 3k_1/(1+\alpha_1)$, then it follows that $\dot{V}_2 < 0$ on $V_{20} \leq V_{M0}$, where $V_{20} = k_1\sum_{i=1}^3 |q_i|^{1+\alpha_1}/(1+\alpha_1) + \hat{\omega}^T J\hat{\omega}/2 + V_\alpha(\zeta)$, which implies that all signals in the closed-loop system are bounded.

If $V_\alpha(\zeta) < 1$, then (57) becomes

$$\begin{aligned}
\dot{V}_2 &\leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - c_2K_2V_\alpha^\beta(\zeta) + (c_3+2)K_2V_\alpha^{\alpha_1/2}(\zeta) \\
&= -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) - K_2V_\alpha^{\alpha_1/2}(\zeta)\left[c_2V_\alpha^{\beta-\alpha_1/2}(\zeta) - (c_3+2)\right]
\end{aligned} \tag{59}$$

and it follows that $\dot{V}_2 < 0$ when $V_\alpha(\zeta) > [(c_3+2)/c_2]^{1/(\beta-\alpha_1/2)}$. Therefore, we conclude that all signals in the closed-loop system are bounded, and there exists a positive constant Δ such that $\|\omega\| \leq \Delta$ and $\|\hat{\omega}\| \leq \Delta$. From Theorem 1, we claim that $\tilde{\bar{q}}$ and $\tilde{\omega}$ converge to zero in a finite time t_{f1}. When $t \geq t_{f1}$, V_2 and $\dot{V}_2$ become

$$V_2 = \frac{k_1}{1+\alpha_1}\sum_{i=1}^3 |q_i|^{1+\alpha_1} + \frac{1}{2}\hat{\omega}^T J\hat{\omega} \tag{60}$$

$$\dot{V}_2 \leq -k_2\hat{\omega}^T \mathrm{sat}_{\alpha_2}(\hat{\omega}) \leq 0. \tag{61}$$

Moreover, when $t \geq t_{f1}$, $\bar{q}$ and $\hat{\omega}$ are governed by the following dynamic equations:

$$\begin{cases} \dot{\bar{q}} = M_1\hat{\omega} \\ J\dot{\hat{\omega}} = -\hat{\omega}^\times J\hat{\omega} - k_1M_1^T\mathrm{sig}^{\alpha_1}(\bar{q}) - k_2\mathrm{sat}_{\alpha_2}(\hat{\omega}). \end{cases} \tag{62}$$

To prove the above system is finite-time stable, two steps are considered as follows.

Step (1): Asymptotic Stability of System (62).

Notice that $\dot{V}_2 \equiv 0$ implies $\hat{\omega} \equiv 0$, which, in turn, implies $\bar{q} \equiv 0$ using (62) and the fact that M_1 is nonsingular when $(\bar{q}, \hat{\omega}) \in \Omega_3$. By LaSalle's invariant set theorem, the zero solution of system (62) is asymptotically stable, i.e., $\bar{q}$ and $\hat{\omega}$ converge to zero as $t \to \infty$, which implies that q_4 converges to $+1$ or -1 as $t \to \infty$.

Step (2): Local Finite-Time Stability of System (62).

In this step, we show that system (62) is locally finite-time stable. System (62) can be rewritten as:

$$\begin{cases} \dot{\overline{q}} = \frac{s}{2}\hat{\omega} + f_3(\overline{q},\hat{\omega}) \\ \dot{\hat{\omega}} = -\frac{sk_1}{2}J^{-1}\mathrm{sig}^{\alpha_1}(\overline{q}) - k_2J^{-1}\mathrm{sig}^{\alpha_2}(\hat{\omega}) + f_4(\overline{q},\hat{\omega}) \end{cases} \tag{63}$$

where $f_3 = (M_1 - sI_3/2)\,\hat{\omega}$, $f_4 = -J^{-1}\left[\hat{\omega}^\times J\hat{\omega} + k_1\left(M_1^T - sI_3/2\right)\mathrm{sig}^{\alpha_1}(\overline{q})\right]$, and $s = 1$ or $s = -1$ is a constant.

First, we show that the nominal system of (63), i.e., system

$$\begin{cases} \dot{\overline{q}} = \frac{s}{2}\hat{\omega} \\ \dot{\hat{\omega}} = -\frac{sk_1}{2}J^{-1}\mathrm{sig}^{\alpha_1}(\overline{q}) - k_2J^{-1}\mathrm{sig}^{\alpha_2}(\hat{\omega}) \end{cases} \tag{64}$$

is asymptotically stable and homogeneous. Consider the following Lyapunov function:

$$V_2 = \frac{k_1}{1+\alpha_1}\sum_{i=1}^{3}|q_i|^{1+\alpha_1} + \frac{1}{2}\hat{\omega}^T J\hat{\omega} \tag{65}$$

Taking the time derivative of V_2 along with system (64) results in

$$\dot{V}_2 = -k_2\hat{\omega}^T\mathrm{sig}^{\alpha_2}(\hat{\omega}) \le 0. \tag{66}$$

Following the similar proof as that in Step (1) and using LaSalle's invariant set theorem, we conclude that the solution of system (64) is asymptotically stable. Define the dilation $\delta_\lambda^r(\overline{q},\hat{\omega}) = (\lambda\overline{q}^T, \lambda^\alpha\hat{\omega}^T)$, and we can verify that system (64) is homogeneous of degree $\alpha - 1 < 0$ with respect to the dilation $\delta_\lambda^r(\overline{q},\hat{\omega})$. Furthermore, by noting that $q_4 = s\sqrt{1-\|\overline{q}\|^2}$, f_3 and f_4 can be reexpressed as

$$f_3 = \frac{s(\sqrt{1-\|\overline{q}\|^2}-1)I_3 + \overline{q}^\times}{2}\hat{\omega} \tag{67}$$

$$\begin{aligned} f_4 &= -J^{-1}\hat{\omega}^\times J\hat{\omega} - k_1J^{-1}\left(M_1^T - sI_3/2\right)\mathrm{sig}^{\alpha_1}(\overline{q}) \\ &= -J^{-1}\hat{\omega}^\times J\hat{\omega} - \frac{sk_1}{2}J^{-1}\left((\sqrt{1-\|\overline{q}\|^2}-1)I_3 - \overline{q}^\times\right)\mathrm{sig}^{\alpha_1}(\overline{q}) \end{aligned} \tag{68}$$

and it can be verified that

$$\lim_{\lambda\to 0}\frac{f_3(\lambda\overline{q},\lambda^\alpha\hat{\omega})}{\lambda^\alpha} = \lim_{\lambda\to 0}\left(\frac{-s\lambda^2\|\overline{q}\|^2\hat{\omega}}{2(1+\sqrt{1-\lambda^2\|\overline{q}\|^2})} + \frac{\lambda\overline{q}^\times\hat{\omega}}{2}\right) = 0 \tag{69}$$

$$\lim_{\lambda\to 0}\frac{f_4(\lambda\overline{q},\lambda^\alpha\hat{\omega})}{\lambda^{2\alpha-1}} = \lim_{\lambda\to 0}\left(-\lambda J^{-1}\hat{\omega}^\times J\hat{\omega} + \frac{\lambda sk_1}{2}J^{-1}\overline{q}^\times\mathrm{sig}^{\alpha_1}(\overline{q}) + \frac{sk_1\lambda^2J^{-1}\|\overline{q}\|^2\mathrm{sig}^{\alpha_1}(\overline{q})}{2(1+\sqrt{1-\lambda^2\|\overline{q}\|^2})}\right) = 0. \tag{70}$$

Using Lemma 5, we conclude that system (63) is locally finite-time stable. By the results of Steps (1) and (2), we obtain that the zero solution of system (62) is finite-time stable, i.e., $\overline{q}$ and $\hat{\omega}$ converge to zero in finite time. Since $\omega(t) = \hat{\omega}(t)$ when $t \ge t_{f1}$, we conclude that ω also converges to zero in finite time.

Remark 6. Note that there is a switch in the control law (47). It is important to point out if $t \ge t_{f1}$ and $(\overline{q},\hat{\omega}) \in \Omega_3$, then we obtain that $\dot{V}_2 \le 0$, which implies that Ω_3 is an invariant set; that is, once $t \ge t_{f1}$ and $(\overline{q},\hat{\omega}) \in \Omega_3$, then $(\overline{q},\hat{\omega})$ will stay within the set Ω_3 forever and there will be no more switching between two controllers.

Remark 7. In practice, there always exists measurement noise which may cause pathological problems when control laws involve switching. Specifically, it may be possible for a certain choice of measurement error to get stuck in a hysteresis type loop around the switching point. For this case, the measured quaternion $q_m = q + q_n$ is used to replace the true quaternion q in the control law, where q_m is the measured quaternion and q_n represents the measurement noise, and we modify the switch rule to overcome this problem. Assume the magnitude of the noise measurement is $|q_{ni}| \leq n_m (i = 1, 2, 3, 4)$, where n_m is a known positive constant. Let $g_i(q_{ni}) = |q_i + q_{ni}|^{1+\alpha_1}$. By mean value theorem, we have

$$\begin{aligned} g_i &= |q_i|^{1+\alpha_1} + \frac{\partial g_i}{\partial q_{ni}}(q_{vi})q_{ni} \leq |q_i|^{1+\alpha_1} + (1+\alpha_1)(1+|q_{ni}|)^{\alpha_1}|q_{ni}| \\ &\leq |q_i|^{1+\alpha_1} + n_m(1+\alpha_1)(1+n_m)^{\alpha_1} \end{aligned} \tag{71}$$

where q_{vi} is some point in $(0, q_{ni})$, which implies that $|q_{mi}|^{1+\alpha_1} = |q_i + q_{ni}|^{1+\alpha_1} \leq |q_i|^{1+\alpha_1} + n_m(1+\alpha_1)(1+n_m)^{\alpha_1}$. Similarly, we have $|q_{mi}|^{1+\alpha_1} = |q_i + q_{ni}|^{1+\alpha_1} \geq |q_i|^{1+\alpha_1} - n_m(1+\alpha_1)(1+n_m)^{\alpha_1}$. The switch rule is now modified as: if

$$\sum_{i=1}^{3} |q_{mi}|^{1+\alpha_1} + \frac{1+\alpha_1}{2k_1}\hat{\omega}^T J\hat{\omega} > 1 + 3n_m(1+\alpha_1)(1+n_m)^{\alpha_1} \tag{72}$$

then the control law is $\tau = -k_1\bar{q} - k_2\text{sat}(\hat{\omega})$; if

$$\sum_{i=1}^{3} |q_{mi}|^{1+\alpha_1} + \frac{1+\alpha_1}{2k_1}\hat{\omega}^T J\hat{\omega} < 1 - 3n_m(1+\alpha_1)(1+n_m)^{\alpha_1} \tag{73}$$

then the control law is $\tau = -k_1 M_1^T \text{sig}^{\alpha_1}(\bar{q}) - k_2\text{sat}_{\alpha_2}(\hat{\omega})$; if

$$1 - 3n_m(1+\alpha_1)(1+n_m)^{\alpha_1} \leq \sum_{i=1}^{3} |q_{mi}|^{1+\alpha_1} + \frac{1+\alpha_1}{2k_1}\hat{\omega}^T J\hat{\omega} \leq 1 + 3n_m(1+\alpha_1)(1+n_m)^{\alpha_1} \tag{74}$$

then the control law at previous time is applied when $t > 0$, and the control law is $\tau = -k_1 M_1^T \text{sig}^{\alpha_1}(\bar{q}) - k_2\text{sat}_{\alpha_2}(\hat{\omega})$ when $t = 0$.

Remark 8. It is worth mentioning that during the transient phase the finite-time controller may require a slightly larger control effort than the asymptotic controller. This can be explained as follows: for any $0 < |x| < 1$, we have $|x|^\alpha > |x|$ with $0 < \alpha < 1$. However, since the finite-time control law can provide a faster transient response and a higher accuracy control performance, during the steady-state stage, the finite-time controller can reduce the energy consumption as compared with the asymptotic controller.

4. Velocity-Free Finite-Time Attitude Tracking Control of Spacecraft

In this section, we consider finite-time attitude tracking control of spacecraft using only attitude measurements. Consider the desired frame B_D with orientation $q_d = [\bar{q}_d^T, q_{d4}]^T$, where the desired attitude q_d is generated by

$$\dot{q}_d = \frac{1}{2}A(q_d)\omega_d \tag{75}$$

with $\omega_d \in R^3$ being the desired angular velocity. The attitude tracking error is defined as $q_e = q_d^{-1} \odot q$, and the angular velocity error is defined as $\omega_e = \omega - C_{q_e}\omega_d$, where $C_{q_e} = C(q_e)$ denotes the corresponding direction cosine matrix relate to the quaternion q_e. Then, the governing differential equations for the attitude tracking error q_e and angular velocity error ω_e are stated as follows:

$$\dot{q}_e = \frac{1}{2}A(q_e)\omega_e \tag{76}$$

$$\begin{aligned} J\dot{\omega}_e &= -\omega^\times J\omega - J(C_{q_e}\dot{\omega}_d - \omega_e^\times C_{q_e}\omega_d) + \tau \\ &= -(\omega_e + C_{q_e}\omega_d)^\times J(\omega_e + C_{q_e}\omega_d) - J(C_{q_e}\dot{\omega}_d - \omega_e^\times C_{q_e}\omega_d) + \tau. \end{aligned} \tag{77}$$

Assumption 1. The desired angular velocity ω_d and its first two derivatives are assumed to be bounded such that $\|\omega_d\| \leq B_1, \|\dot{\omega}_d\| \leq B_2, \|\ddot{\omega}_d\| \leq B_3$, where B_1, B_2, and B_3 are known positive constants satisfying $(\lambda_{max}^2(J) - \lambda_{min}^2(J))^{1/2}B_1^2 + \lambda_{max}(J)B_2 = B_4 < \tau_M$.

Assumption 1 is reasonable because it implies that there exists a control input satisfying the input magnitude constraints such that the attitude of the spacecraft can track the given time-varying reference trajectory.

4.1. Finite-Time Observer

For the attitude tracking problem, we propose the following finite-time observer:

$$\dot{\hat{q}}_e = \frac{1}{2}A(\hat{q}_e)C_{\tilde{q}_e}^{-1}\left[\hat{\omega}_e + \theta\gamma_1 P_1^{-1}\mathrm{sig}^{\alpha}(\bar{\tilde{q}}_e) + \frac{2\gamma_3\bar{\tilde{q}}_e}{\tilde{q}_{e4}(1-\bar{\tilde{q}}_e^T\bar{\tilde{q}}_e)}\right] \tag{78}$$

$$J\dot{\hat{\omega}}_e = -(\hat{\omega}_e + C_{q_e}\omega_d)^{\times}J(\hat{\omega}_e + C_{q_e}\omega_d) - J(C_{q_e}\dot{\omega}_d - \hat{\omega}_e^{\times}C_{q_e}\omega_d) + \tau + \theta^2\gamma_2 J\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}_e) \tag{79}$$

where $\hat{q}_e$ and $\hat{\omega}_e$ denote estimates of q_e and ω_e, respectively, $\hat{q}_e(0) = q_e(0)$, $\hat{\omega}_e(0) = 0$, $\tilde{q}_e = \hat{q}_e^{-1}\odot q_e$ is the quaternion error between q_e and $\hat{q}_e$, $\bar{\tilde{q}}_e = [\tilde{q}_{e1}, \tilde{q}_{e2}, \tilde{q}_{e3}]^T$, $C_{\tilde{q}_e} = C(\tilde{q}_e)$ is the corresponding direction cosine matrix relate to the quaternion $\tilde{q}_e$, and $P_1 = (\tilde{q}_{e4}I_3 + \bar{\tilde{q}}_e^{\times})/2$. Then, the dynamic equations for observer errors $\tilde{q}_e$ and $\tilde{\omega}_e = \omega_e - \hat{\omega}_e$ are given as follows:

$$\dot{\tilde{q}}_e = \frac{1}{2}A(\tilde{q}_e)\left[\tilde{\omega}_e - \theta\gamma_1 P_1^{-1}\mathrm{sig}^{\alpha}(\bar{\tilde{q}}_e) - \frac{2\gamma_3\bar{\tilde{q}}_e}{\tilde{q}_{e4}(1-\bar{\tilde{q}}_e^T\bar{\tilde{q}}_e)}\right] \tag{80}$$

$$\begin{aligned}\dot{\tilde{\omega}}_e &= -J^{-1}\left[(\omega_e + C_{q_e}\omega_d)^{\times}J(\omega_e + C_{q_e}\omega_d) + J(C_{q_e}\dot{\omega}_d - \omega_e^{\times}C_{q_e}\omega_d)\right]\\ &\quad + J^{-1}\left[(\hat{\omega}_e + C_{q_e}\omega_d)^{\times}J(\hat{\omega}_e + C_{q_e}\omega_d) + J(C_{q_e}\dot{\omega}_d - \hat{\omega}_e^{\times}C_{q_e}\omega_d)\right] - \theta^2\gamma_2\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}_e)\\ &= -J^{-1}\left[\tilde{\omega}_e^{\times}J\tilde{\omega}_e + \hat{\omega}_e^{\times}J\tilde{\omega}_e + \tilde{\omega}_e^{\times}J\hat{\omega}_e + \tilde{\omega}_e^{\times}JC_{q_e}\omega_d + (C_{q_e}\omega_d)^{\times}J\tilde{\omega}_e + J\tilde{\omega}_e^{\times}C_{q_e}\omega_d\right] - \theta^2\gamma_2\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}_e)\end{aligned} \tag{81}$$

and the governing equations for the observer errors $\bar{\tilde{q}}_e$ and $\tilde{\omega}_e$ can be expressed as

$$\begin{cases}\dot{\bar{\tilde{q}}}_e = \frac{1}{2}\zeta_2 - \gamma_1\mathrm{sig}^{\alpha}(\bar{\tilde{q}}_e) + f_5(\tilde{q}_e,\ \tilde{\omega}_e)\\ \dot{\tilde{\omega}}_e = -\gamma_2\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}_e) + f_6(\tilde{\omega}_e,\ \hat{\omega}_e,\ q_e,\ \omega_d)\end{cases} \tag{82}$$

where $f_5(\tilde{q}_e,\ \tilde{\omega}_e) = (P_1 - I_3/2)\tilde{\omega}_e - 2\gamma_3P_1\bar{\tilde{q}}_e/\left[\tilde{q}_{e4}(1-\bar{\tilde{q}}_e^T\bar{\tilde{q}}_e)\right]$ and $f_6(\tilde{\omega}_e,\ \hat{\omega}_e,\ q_e,\ \omega_d) = -J^{-1}[\tilde{\omega}_e^{\times}J\tilde{\omega}_e + \hat{\omega}_e^{\times}J\tilde{\omega}_e + \tilde{\omega}_e^{\times}J\hat{\omega}_e + \tilde{\omega}_e^{\times}JC_{q_e}\omega_d + (C_{q_e}\omega_d)^{\times}J\tilde{\omega}_e + J\tilde{\omega}_e^{\times}C_{q_e}\omega_d]$.

Corollary 4. For any positive constant Δ, if $\hat{q}_e$ is generated by (78), $\|\omega_e\| \le \Delta$, and $\|\hat{\omega}_e\| \le \Delta$, then $\|\bar{\tilde{q}}_e(t)\| \le \Delta_1, \forall t \in [0,\infty)$, where $\Delta_1 = \sqrt{\Delta^2/(\Delta^2 + \theta\gamma_1\gamma_3)} < 1$.

Proof. The proof is similar to that of Corollary 2, and thus is omitted here.

Theorem 3. Consider the observer defined by (78) and (79). For any positive constant Δ, if $\|\omega_e\| \le \Delta$, $\|\hat{\omega}_e\| \le \Delta$, and the parameter θ is chosen to be sufficiently large, then the zero solution of system (82) is locally finite-time stable.

Proof. Consider the following coordinate transformation: $\zeta_1 = \bar{\tilde{q}}_e/\theta$ and $\zeta_2 = \tilde{\omega}_e/\theta^2$. Then, we have

$$\begin{cases}\dot{\zeta}_1 = \frac{\theta}{2}\zeta_2 - \theta^{\alpha}\gamma_1\mathrm{sig}^{\alpha}(\zeta_1) + \frac{f_5(\tilde{q}_e,\ \tilde{\omega}_e)}{\theta}\\ \dot{\zeta}_2 = -\theta^{\alpha_1}\gamma_2\mathrm{sig}^{\alpha_1}(\zeta_1) + \frac{f_6(\tilde{\omega}_e,\ \hat{\omega}_e,\ q_e,\ \omega_d)}{\theta^2}\end{cases} \tag{83}$$

Define $\zeta = [\zeta_1^T, \zeta_2^T]^T$, $\chi = [\zeta_1^T, (\mathrm{sig}^{1/(\alpha)}(\zeta_2))^T]^T$, and

$$V_\alpha(\zeta) = \chi^T N\chi. \tag{84}$$

The time derivative of $V_\alpha(\zeta)$ along with system (83) is given by

$$\dot{V}_\alpha(\zeta) \le -c_2(\alpha,\theta)V_\alpha^{\beta}(\zeta) + 2\chi^T N\begin{bmatrix}\frac{1}{\theta}f_5\\ \frac{1}{\alpha\theta^2}\mathrm{diag}\left(|\zeta_2|^{\frac{1}{\alpha}-1}\right)f_6\end{bmatrix} \tag{85}$$

If we choose the observer parameter θ such that $\theta \ge \Delta^2$, then we have $\|\bar{\tilde{q}}_e(t)\| \le \Delta_1 \le \Delta_2 = \sqrt{1/(1+\gamma_1\gamma_3)} < 1$ and $\tilde{q}_{e4}(t) \ge \sqrt{1-\Delta_2^2}, \forall t \in [0,\infty)$, where Lemma 10 has been applied. Noting that

$$\|f_5\| \le \left[2\Delta + \frac{\sqrt{2}\gamma_3}{(1-\Delta_2^2)^{3/2}}\right]\|\bar{\tilde{q}}_e\| = \rho_1\|\bar{\tilde{q}}_e\| \tag{86}$$

$$\|f_6\| \le \lambda_{\max}(J^{-1})\lambda_{\max}(J)(4\Delta + 3B_1)\|\tilde{\omega}_e\| = \rho_2\|\tilde{\omega}_e\| \tag{87}$$

we obtain

$$2\chi^T N \begin{bmatrix} \frac{1}{\theta} f_5 \\ \frac{1}{\alpha\theta^2}\mathrm{diag}\left(|\zeta_2|^{\frac{1}{\alpha}-1}\right) f_6 \end{bmatrix} \leq 2\lambda_{\max}(N)\|\chi\| \left[\rho_1\|\chi_1\| + \frac{\rho_2}{\alpha}\left(\sum_{i=1}^{3}|\zeta_{2i}|^{\frac{1}{\alpha}-1}\right)\left(\sum_{i=1}^{3}|\zeta_{2i}|\right)\right]. \tag{88}$$

Following the same procedure as the proof of Theorem 1, we can obtain that

$$\dot{V}_\alpha(\zeta) \leq -c_2 V_\alpha^\beta(\zeta) + c_3 V_\alpha(\zeta) \tag{89}$$

and the origin of system (83) is locally finite-time stable, and consequently, the zero solution of system (82) is locally finite time.

4.2. Finite-Time Attitude Tracking Controller Design

The finite-time output feedback attitude tracking controller is chosen as

$$\tau = \begin{cases} (C_{q_e}\omega_d)^\times J C_{q_e}\omega_d + J C_{q_e}\dot{\omega}_d - k_3 M_2^T \mathrm{sig}^{\alpha_1}(\bar{q}_e) - k_4 \mathrm{sat}_{\alpha_2}(\hat{\omega}_e), & \text{if} \quad (\bar{q}_e, \hat{\omega}_e) \in \Omega_4 \\ (C_{q_e}\omega_d)^\times J C_{q_e}\omega_d + J C_{q_e}\dot{\omega}_d - k_3\bar{q}_e - k_4\mathrm{sat}(\hat{\omega}_e), & \text{otherwise} \end{cases} \tag{90}$$

where k_3 and k_4 are positive constants, $M_2 = (q_{e4}I_3 + \bar{q}_e^\times)/2$, and Ω_4 is defined by

$$\Omega_4 = \left\{(\bar{q}_e, \hat{\omega}_e) \left| \sum_{i=1}^{3}|q_{ei}|^{1+\alpha_1} + \frac{1+\alpha_1}{2k_3}\hat{\omega}_e^T J \hat{\omega}_e < 1 \right.\right\}. \tag{91}$$

In view of Lemma 6 and Assumption 1, the bound on $\tau_i (i = 1, 2, 3)$ is $|\tau_i| \leq B_4 + k_3 + k_4$. We can choose the control gains k_3 and k_4 such that $B_4 + k_3 + k_4 \leq \tau_M$, which implies that the control torque defined by (90) always satisfies the input constraints.

Lemma 8. Consider the spacecraft described by (2) and (3)

$$\tau = (C_{q_e}\omega_d)^\times J C_{q_e}\omega_d + J C_{q_e}\dot{\omega}_d - k_3\bar{q}_e - k_4\mathrm{sat}(\hat{\omega}_e) \tag{92}$$

where $\hat{\omega}_e$ is generated by the observer defined by (78) and (79). For any positive constant V_M, if the initial conditions satisfy $V_{30}(q_{e4}(0), \hat{\omega}_e(0), \zeta(0)) \leq V_M$, where V_{30} is defined by

$$V_{30} = 2k_3(1 - q_{e4}) + \frac{1}{2}\hat{\omega}_e^T J \hat{\omega}_e + V_\alpha(\zeta) \tag{93}$$

and $V_\alpha(\zeta)$ is given by (84), then there exists a sufficiently large observer parameter θ such that $\bar{q}_e$ and ω_e converge to zero as $t \to \infty$.

Proof. Substituting the control law (92) into (79) results in

$$\begin{aligned} J\dot{\hat{\omega}}_e &= -\hat{\omega}_e^\times J\hat{\omega}_e - \hat{\omega}_e^\times J C_{q_e}\omega_d - (C_{q_e}\omega_d)^\times J\hat{\omega}_e + J\hat{\omega}_e^\times C_{q_e}\omega_d + \theta^2\gamma_2 J\mathrm{sig}^{\alpha_1}(\tilde{\bar{q}}_e) - k_3\bar{q}_e - k_4\mathrm{sat}(\hat{\omega}_e) \\ &= -\hat{\omega}_e^\times J\hat{\omega}_e - \hat{\omega}_e^\times J C_{q_e}\omega_d - \left[(C_{q_e}\omega_d)^\times J + J(C_{q_e}\omega_d)^\times\right]\hat{\omega}_e + \theta^2\gamma_2 J\mathrm{sig}^{\alpha_1}(\tilde{\bar{q}}_e) - k_3\bar{q}_e - k_4\mathrm{sat}(\hat{\omega}_e). \end{aligned} \tag{94}$$

Noticing that

$$\hat{\omega}_e^T J\dot{\hat{\omega}}_e = \theta^2\gamma_2\hat{\omega}_e^T J\mathrm{sig}^{\alpha_1}(\tilde{\bar{q}}_e) - k_3\hat{\omega}_e^T\bar{q}_e - k_4\hat{\omega}_e^T\mathrm{sat}(\hat{\omega}_e) \tag{95}$$

and following the similar analysis as that in the proof of Lemma 7, we can conclude that $\bar{q}_e$ and ω_e converge to zero as $t \to \infty$.

Theorem 4. Consider the spacecraft described by (2) and (3) under the control law (90), where $\hat{\omega}_e$ is generated by the observer given by (78) and (79). For any positive constant V_M, if the initial conditions satisfy $V_{30}(q_{e4}(0), \hat{\omega}_e(0), \zeta(0)) \leq V_M$, where V_{30} is defined by (93), then there exists a sufficiently large observer parameter θ such that $\bar{q}_e$ and ω_e converge to zero in finite time.

Proof. Using the fact that

$$-\hat{\omega}_e^T \hat{\omega}_e^\times J \hat{\omega}_e - \hat{\omega}_e^T \hat{\omega}_e^\times J C_{q_e} \omega_d - \hat{\omega}_e^T \left[(C_{q_e} \omega_d)^\times J + J (C_{q_e} \omega_d)^\times \right] \hat{\omega}_e = 0 \tag{96}$$

and following the similar procedure as that in the proof of Theorem 2, we can obtain that $\bar{q}_e$ and ω_e converge to zero in finite time.

Remark 9. The velocity-free attitude control problem has also been studied in [24, 25, 26, 27, 28, 29, 30]. It is worth pointing out that asymptotic stability is achieved in these works. Nevertheless, the main goal of the present work is to design finite-time output feedback attitude control laws. As compared with the asymptotic controller, the finite-time controller can provide faster convergence rate, higher precision control performance and better disturbance rejection property.

5. Simulation Results

In this section, numerical simulations are presented to verify the effectiveness of the proposed controller. The inertia matrix is chosen as $J = [20\ 1.2\ 0.9; 1.2\ 17\ 1.4; 0.9\ 1.4\ 15] kg \cdot m^2$ [24]. The observer parameters are chosen as: $\gamma_1 = \gamma_2 = \gamma_3 = 2, \theta = 10$, and $\alpha = 0.9$. In addition, the limit on the input magnitude is considered to be $|\tau_i| \leq 10 (i = 1, 2, 3)$ Nm.

5.1. Attitude Stabilizing Control for Spacecraft

For the attitude stabilization, the controller parameters are set as: $k_1 = k_2 = 5$. The initial conditions are considered as: $\bar{q}(0) = [-0.6, 0.4, -0.2]^T$, $q_4(0) = \sqrt{1 - \|\bar{q}(0)\|^2}$, and $\omega(0) = [1.2, -1.5, 0.2]^T$. With the control law (47), the results are shown in Fig. 1. Obviously, a satisfactory attitude performance is achieved by using the finite-time attitude controller (47) even in the absence of angular velocity measurements.

Next, the performance of the finite-time controller (47) is compared with the asymptotic controller presented in [28]. The parameters of the asymptotic controller are chosen as: $a_1 = a_2 = 5$, and $\Gamma_1 = 3$. The initial conditions are considered as: $q(0) = [1, 0, 0, 0]^T$, and $\omega(0) = [1.2, -1.5, 0.2]^T$. The response of $\|\bar{q}\|$ and $\|\tau\|$ is show in Fig. 2. It is found that the performance of the proposed finite-time controller is much better than the asymptotic controller in [28]. In order to improve the performance of the asymptotic controller, the control gains are increased as $a_1 = a_2 = 20$. The results are depicted in Figs. 3-4. It is found that the performance of the proposed finite-time controller is still better than the asymptotic controller in [28]. It is important to mention that the gains of the asymptotic controller are much larger than the gains used in the proposed finite-time controller and the value of the applied control torque may exceed the limit on the input magnitude as shown in Fig. 4.

The effect of the external disturbance on the performance of the controller is examined. In this case, the disturbance is assumed as $d = 0.01[\sin(t), \cos(t), \sin(2t)]^T$. From Fig. 5, it is observed that, although the gains of the asymptotic controller in [28] are chosen as large as 20, the proposed finite-time output feedback controller can still provide a better disturbance rejection property than the asymptotic controller.

Next, the effect of the measurement noise on the performance of the controller is examined. The measured quaternion is assumed to be $q_m = q + q_n$, where q_m is the measured quaternion, q is the true quaternion, and $q_n = 0.01[\sin(t), \cos(t), \sin(2t), -\cos(2t)]^T$ represents the measurement noise. Furthermore, we assume that there is no external disturbance acting on the spacecraft and that the input limit is $|\tau_i| \leq 10 (i = 1, 2, 3)$ Nm. In this case, the normalized measured quaternion $q_m / \|q_m\|$ is used in the observer, the measured quaternion q_m is used in the control law, and we set $\hat{q}(0) = q_m(0) / \|q_m(0)\|$. The response of $\|\bar{q}\|$ and $\|\tau\|$ for the proposed controller and the asymptotic controller in [28] is depicted in Fig. 6. It is observed that the proposed controller requires much less energy consumption but provide a higher accuracy attitude performance as compared to the asymptotic controller.

5.2. Attitude Tracking Control for Spacecraft

For the attitude tracking, the controller parameters are selected as: $k_3 = k_4 = 3$. The initial conditions are set as: $\bar{q}(0) = [0.6, -0.4, 0.2]^T$, $q_4(0) = \sqrt{1 - \|\bar{q}(0)\|^2}$, and $\omega(0) = [1, -2, 1.5]^T$. The reference attitude q_d is generated by (75) with the desired angular velocity $\omega_d = 0.1[\sin(0.2\pi t), \sin(0.2\pi t), \sin(0.2\pi t)]^T$ and $q_d(0) = [0, 0, 0, 1]^T$. The velocity-free finite-time control law (90) demonstrates a good attitude tracking performance as shown in Fig. 7.

6. Conclusions

Based on a finite-time observer and the homogeneous method, two quaternion-based velocity-free finite-time attitude control laws were proposed for rigid spacecraft subject to control input saturation. The performance of the proposed method was demonstrated through numerical simulations of the governing nonlinear system equations of motion and was compared with an asymptotic output feedback control law from the literature. It was shown that the proposed finite-time controller can provide a faster convergence rate, better disturbance rejection property, and higher accuracy control performance than the asymptotic controller. One of our future work is to extend the results in this paper to attitude coordination control of a group of spacecraft.

Appendix

Proof of Corollary 1

Note that $U_1 \subseteq U$. If $x \in U_1$, then we have

$$\dot{V}(x,t) \leq -lV^a(x,t) + kV(x,t) = -(l - kV^{1-a}(x,t))V^a(x,t) \leq 0 \tag{97}$$

which implies that $V(x,t) \leq V(x(t_0),t_0)$ for any initial condition $x(t_0) \in U_1$. Thus, (97) becomes

$$\dot{V}(x,t) \leq -(l - kV^{1-a}(x(t_0),t_0))V^a(x,t). \tag{98}$$

By Lemma 1, it is concluded that the origin of system (9) is locally finite-time stable and the settling time satisfies (14).

Proof of Lemma 3

First, we show that

$$\left(\sum_{i=1}^{n} |x_i|\right)^{\nu} \leq \sum_{i=1}^{n} |x_i|^{\nu}. \tag{99}$$

We consider two cases, i.e., $\sum_{i=1}^n |x_i| = 0$ and $\sum_{i=1}^n |x_i| \neq 0$. When $\sum_{i=1}^n |x_i| = 0$, it is apparent that (99) is satisfied. When $\sum_{i=1}^n |x_i| \neq 0$, by noting that $\nu \in (0,1]$, we have

$$\sum_{i=1}^{n} \left(\frac{|x_i|}{\sum_{j=1}^n |x_j|}\right)^{\nu} \geq \sum_{i=1}^{n} \frac{|x_i|}{\sum_{j=1}^n |x_j|} = 1 \tag{100}$$

which implies that the inequality (99) holds.

Next, we show that

$$\sum_{i=1}^{n} |x_i|^{\nu} \leq n^{1-\nu}\left(\sum_{i=1}^{n} |x_i|\right)^{\nu}. \tag{101}$$

We also consider two cases, i.e., $\sum_{i=1}^n |x_i| = 0$ and $\sum_{i=1}^n |x_i| \neq 0$. When $\sum_{i=1}^n |x_i| = 0$, it is easy to verify that (101) is satisfied. When $\sum_{i=1}^n |x_i| \neq 0$ and $\nu = 1$, it is clear that the inequality (101) holds. When $\sum_{i=1}^n |x_i| \neq 0$ and $\nu \in (0,1)$, by Lemma 2, we obtain

$$\frac{1}{n^{1-\nu}}\left(\frac{|x_i|}{\sum_{j=1}^n |x_j|}\right)^{\nu} \leq \frac{1-\nu}{n} + \frac{\nu|x_i|}{\sum_{j=1}^n |x_j|} \tag{102}$$

$$\sum_{i=1}^{n}\frac{1}{n^{1-\nu}}\left(\frac{|x_i|}{\sum_{j=1}^n |x_j|}\right)^{\nu} \leq \sum_{i=1}^{n}\frac{1-\nu}{n} + \frac{\nu\sum_{i=1}^n |x_i|}{\sum_{j=1}^n |x_j|} = 1 \tag{103}$$

which implies that (101) is satisfied. Therefore, we conclude that $(\sum_{i=1}^n |x_i|)^{\nu} \leq \sum_{i=1}^n |x_i|^{\nu} \leq n^{1-\nu}\left(\sum_{i=1}^n |x_i|\right)^{\nu}$.

Proof of Corollary 2

Since $\hat{q}(0) = q(0)$, we have $\tilde{q}(0) = [0, 0, 0, 1]^T$. Defining $V = \bar{\tilde{q}}^T\bar{\tilde{q}}/2$, and taking the time derivative of V along with (21) lead to

$$\begin{aligned}
\dot{V} &= \bar{\tilde{q}}^T P\tilde{\omega} - \theta\gamma_1\bar{\tilde{q}}^T \mathrm{sig}^\alpha(\bar{\tilde{q}}) - \frac{\gamma_3\bar{\tilde{q}}^T\bar{\tilde{q}}}{1-\tilde{q}^T\tilde{q}} \\
&\leq \frac{\theta\gamma_1\bar{\tilde{q}}^T\bar{\tilde{q}}}{2} + \frac{\|P\tilde{\omega}\|^2}{2\theta\gamma_1} - \theta\gamma_1\bar{\tilde{q}}^T\bar{\tilde{q}} - \frac{\gamma_3\bar{\tilde{q}}^T\bar{\tilde{q}}}{1-\tilde{q}^T\tilde{q}} \\
&\leq -\frac{\theta\gamma_1\bar{\tilde{q}}^T\bar{\tilde{q}}}{2} - \frac{\gamma_3\bar{\tilde{q}}^T\bar{\tilde{q}}}{1-\tilde{q}^T\tilde{q}} + \frac{\Delta^2}{\theta\gamma_1} \leq \frac{-\left(\gamma_3 + \frac{\Delta^2}{\theta\gamma_1}\right)\bar{\tilde{q}}^T\bar{\tilde{q}} + \frac{\Delta^2}{\theta\gamma_1}}{1-\tilde{q}^T\tilde{q}}
\end{aligned} \tag{104}$$

where the following inequalities

$$-\theta\gamma_1\bar{\tilde{q}}^T \mathrm{sig}^\alpha(\bar{\tilde{q}}) \leq -\theta\gamma_1\bar{\tilde{q}}^T\bar{\tilde{q}} \tag{105}$$

$$\|P\| \leq \frac{1}{2}(|\tilde{q}_4| + \|\bar{\tilde{q}}\|) \leq \frac{1}{2}\sqrt{2(|\tilde{q}_4|^2 + \|\bar{\tilde{q}}\|^2)} = \frac{\sqrt{2}}{2} \tag{106}$$

are applied. Thus, $\dot{V}$ is strictly negative outside the following compact set $\Omega_{\bar{\tilde{q}}}$:

$$\Omega_{\bar{\tilde{q}}} = \left\{\bar{\tilde{q}}\,\middle|\, 0 \leq \|\bar{\tilde{q}}\| \leq \sqrt{\frac{\Delta^2}{\Delta^2 + \theta\gamma_1\gamma_3}} = \Delta_1 < 1\right\} \tag{107}$$

which implies that $\|\bar{\tilde{q}}\|$ decreases whenever $\bar{\tilde{q}}$ is outside the compact set $\Omega_{\bar{\tilde{q}}}$. Furthermore, since $\|\bar{\tilde{q}}(0)\| = 0$, we conclude that $\|\bar{\tilde{q}}(t)\| \leq \Delta_1$ for all $0 \leq t < \infty$.

Proof of Corollary 3

The analysis procedure is similar to that of the proof of Theorem 1, except for the presence of the disturbances. Considering the Lyapunov function given in (26), and using (37), we obtain

$$\begin{aligned}
\dot{V}_\alpha &\leq -c_2V_\alpha^\beta + c_3V_\alpha + 2\chi^T N\begin{bmatrix} 0 \\ \frac{1}{\alpha\theta^2}\mathrm{diag}\left(|\zeta_2|^{\frac{1}{\alpha}-1}\right)d \end{bmatrix} \\
&\leq -c_2V_\alpha^\beta + c_3V_\alpha + \frac{2d_M\lambda_{\max}(N)}{\alpha\theta^2}\|\chi\|\sum_{i=1}^{3}|\zeta_{2i}|^{\frac{1}{\alpha}-1}.
\end{aligned} \tag{108}$$

By Lemma 3, we have

$$\sum_{i=1}^{3}|\zeta_{2i}|^{\frac{1}{\alpha}-1} = \sum_{i=1}^{3}|\chi_{2i}|^{1-\alpha} \leq 3^{\frac{1+\alpha}{2}}\|\chi_2\|^{1-\alpha} \tag{109}$$

and it follows that

$$\begin{aligned}
\dot{V}_\alpha &\leq -c_2V_\alpha^\beta + c_3V_\alpha + \frac{2\times 3^{\frac{1+\alpha}{2}}d_M\lambda_{\max}(N)}{\alpha\theta^2}\|\chi\|^{2-\alpha} \\
&\leq -c_2V_\alpha^\beta + c_3V_\alpha + \frac{2\times 3^{\frac{1+\alpha}{2}}d_M\lambda_{\max}(N)}{\alpha\theta^2(\lambda_{\min}(N))^{\frac{2-\alpha}{2}}}V_\alpha^{(2-\alpha)/2} \\
&= -c_2V_\alpha^\beta + c_3V_\alpha + \frac{c_4}{\theta^2}V_\alpha^{(2-\alpha)/2}.
\end{aligned} \tag{110}$$

Next, we consider two cases for the analysis.

Case 1: $V_\alpha > 1$.

Noting that $V_\alpha > 1$ and $0 < (2-\alpha)/2 < 1$, (110) becomes

$$\dot{V}_\alpha \le -c_2 V_\alpha^\beta + (c_3 + c_4/\theta^2)V_\alpha. \tag{111}$$

Using Corollary 1, we conclude that V_α converges to the region $V_\alpha \le 1$ in finite time.

Case 2: $V_\alpha \le 1$.

Because $V_\alpha \le 1$, $0 < \beta < 1$, and we can choose a sufficiently large parameter θ such that $c_2 > c_3$, we obtain

$$\begin{aligned}\dot{V}_\alpha &\le -(c_2 - c_3)V_\alpha^\beta + \frac{c_4}{\theta^2}V_\alpha^{(2-\alpha)/2}\\ &= -(c_2 - c_3)V_\alpha^{(2-\alpha)/2}\left(V_\alpha^{(2\alpha-1)/2} - \frac{c_4}{(c_2 - c_3)\theta^2}\right)\end{aligned} \tag{112}$$

which implies that $\dot{V}_\alpha < 0$ if $V_\alpha^{(2\alpha-1)/2} > c_4/((c_2 - c_3)\theta^2)$. Thus, we can conclude that V_α converges to the region $V_\alpha \le [c_4/((c_2 - c_3)\theta^2)]^{2/(2\alpha-1)}$ in finite time, i.e., the error χ converges to the region

$$\|\chi\| \le \frac{1}{\sqrt{\lambda_{\min}(N)}}\left(\frac{c_4}{(c_2 - c_3)\theta^2}\right)^{1/(2\alpha-1)} = \Delta_4 \tag{113}$$

in finite time. If $\|\chi\| \le \Delta_4$ and we choose a sufficiently large parameter $\theta > 1$ such that $\Delta_4 < 1$, then the observer error e can be bounded by

$$\begin{aligned}\|e\| &\le \theta^2\left(\sum_{i=1}^3 |\zeta_{1i}| + \sum_{i=1}^3 |\zeta_{2i}|\right) \le 6^{1-\alpha/2}\theta^2\left(\sum_{i=1}^3 |\zeta_{1i}|^{2/\alpha} + \sum_{i=1}^3 |\zeta_{2i}|^{2/\alpha}\right)^{\alpha/2}\\ &\le 6^{1-\alpha/2}\theta^2\|\chi\|^\alpha \le \frac{6^{1-\alpha/2}\theta^2}{\lambda_{\min}^{\alpha/2}(N)}\left(\frac{c_4}{(c_2 - c_3)\theta^2}\right)^{\alpha/(2\alpha-1)}\\ &= \frac{6^{1-\alpha/2}}{\lambda_{\min}^{\alpha/2}(N)}\left(\frac{c_4}{(c_2 - c_3)\theta^{2(1-\alpha)/\alpha}}\right)^{\alpha/(2\alpha-1)} = \Delta_e.\end{aligned} \tag{114}$$

Therefore, we conclude that the observer error e converges to the region $\|e\| \le \Delta_e$ in finite time.

Proof of Lemma 7

Using the control law (50), the dynamic equations for q and $\hat{\omega}$ are expressed as

$$\dot{q} = \frac{1}{2}A(q)\hat{\omega} + \frac{1}{2}A(q)\tilde{\omega} = \frac{1}{2}A(q)\hat{\omega} + \frac{\theta^{1+\theta}}{2}A(q)\zeta_2 \tag{115}$$

$$\begin{aligned}J\dot{\hat{\omega}} &= -\hat{\omega}^\times J\hat{\omega} + \theta^2\gamma_2 J\mathrm{sig}^{\alpha_1}(\bar{\tilde{q}}) - k_1\bar{q} - k_2\mathrm{sat}(\hat{\omega})\\ &= -\hat{\omega}^\times J\hat{\omega} - k_1\bar{q} - k_2\mathrm{sat}(\hat{\omega}) + \theta^{2+\alpha_1}\gamma_2 J\mathrm{sig}^{\alpha_1}(\zeta_1).\end{aligned} \tag{116}$$

Consider the following Lyapunov function:

$$V_1 = 2k_1(1 - q_4) + \frac{1}{2}\hat{\omega}^T J\hat{\omega} + K_1 V_\alpha(\zeta) \tag{117}$$

where $V_\alpha(\zeta)$ is defined by (26), and K_1 is a sufficiently large positive constant.

Taking the time derivative of V_1 along with (37), (115) and (116) yields

$$\begin{aligned}\dot{V}_1 &\le -k_2\hat{\omega}^T\mathrm{sat}(\hat{\omega}) - c_2K_1V_\alpha^\beta(\zeta) + c_3K_1V_\alpha(\zeta) + k_1\theta^2\bar{q}^T\zeta_2 + \theta^{2+\alpha_1}\gamma_2\hat{\omega}^T J\mathrm{sig}^{\alpha_1}(\zeta_1)\\ &\le -k_2\hat{\omega}^T\mathrm{sat}(\hat{\omega}) - c_2K_1V_\alpha^\beta(\zeta) + c_3K_1V_\alpha(\zeta) + 3^{(1-\alpha_1)/2}\gamma_2\theta^{2+\alpha_1}\lambda_{\max}(J)\|\hat{\omega}\|\|\zeta_1\|^{\alpha_1} + k_1\theta^2\|\zeta_2\|\\ &\le -k_2\hat{\omega}^T\mathrm{sat}(\hat{\omega}) - c_2K_1V_\alpha^\beta(\zeta) + c_3K_1V_\alpha(\zeta) + 3^{(1-\alpha)/2}k_1\theta^2\|\chi_2\|^\alpha + 3^{(1-\alpha_1)/2}\gamma_2\theta^{2+\alpha_1}\lambda_{\max}(J)\|\hat{\omega}\|\|\zeta_1\|^{\alpha_1}\\ &\le -k_2\hat{\omega}^T\mathrm{sat}(\hat{\omega}) - c_2K_1V_\alpha^\beta(\zeta) + c_3K_1V_\alpha(\zeta) + \rho_3V_\alpha^{\alpha/2}(\zeta) + \rho_4V_\alpha^{\alpha_1/2}(\zeta)\end{aligned} \tag{118}$$

where the inequalities $\|\hat{\omega}\| \le \sqrt{2V_M/\lambda_{\min}(J)}$, $\|\text{sig}^{\alpha_1}(\zeta_1)\| \le 3^{(1-\alpha_1)/2}\|\zeta_1\|^{\alpha_1}$ and $\|\zeta_2\| \le 3^{(1-\alpha)/2}\|\chi_2\|^{\alpha}$ are used, $\rho_3 = 3^{(1-\alpha)/2}k_1\theta^2/\lambda_{\min}^{\alpha/2}(N)$, and $\rho_4 = 3^{(1-\alpha_1)/2}\sqrt{2V_M}\gamma_2\theta^{2+\alpha_1}\lambda_{\max}(J)/\sqrt{\lambda_{\min}(J)\lambda_{\min}^{\alpha_1}(N)}$. Let $K_1 > \max\{\rho_3, \rho_4\}$, then (118) becomes

$$\dot{V}_1 \le -k_2\hat{\omega}^T\text{sat}(\hat{\omega}) - c_2K_1V_\alpha^\beta(\zeta) + c_3K_1V_\alpha(\zeta) + K_1V_\alpha^{\alpha/2}(\zeta) + K_1V_\alpha^{\alpha_1/2}(\zeta). \tag{119}$$

Next, we consider two cases, i.e., $V_\alpha(\zeta) \ge 1$ and $V_\alpha(\zeta) < 1$. If $V_\alpha(\zeta) \ge 1$, then (119) becomes

$$\begin{aligned}\dot{V}_1 &\le -k_2\hat{\omega}^T\text{sat}(\hat{\omega}) - c_2K_1V_\alpha^\beta(\zeta) + (c_3+2)K_1V_\alpha(\zeta)\\ &= -k_2\hat{\omega}^T\text{sat}(\hat{\omega}) - K_1V_\alpha^\beta(\zeta)\left[c_2 - (c_3+2)V_\alpha^{1-\beta}(\zeta)\right].\end{aligned} \tag{120}$$

We can choose a sufficiently large parameter θ such that $c_2 > (c_3+2)V_M^{1-\beta}$, then it follows that $\dot{V}_1 < 0$ on $V_{10} \le V_M$. This indicates that all signals in the closed-loop system are bounded.

If $V_\alpha(\zeta) < 1$, then (119) becomes

$$\begin{aligned}\dot{V}_1 &\le -k_2\hat{\omega}^T\text{sat}(\hat{\omega}) - c_2K_1V_\alpha^\beta(\zeta) + (c_3+2)K_1V_\alpha^{\alpha_1/2}(\zeta)\\ &= -k_2\hat{\omega}^T\text{sat}(\hat{\omega}) - K_1V_\alpha^{\alpha_1/2}(\zeta)\left[c_2V_\alpha^{\beta-\alpha_1/2}(\zeta) - (c_3+2)\right]\end{aligned} \tag{121}$$

and it follows that $\dot{V}_1 < 0$ when $V_\alpha(\zeta) > [(c_3+2)/c_2]^{1/(\beta-\alpha_1/2)}$. Therefore, we conclude that all signals in the closed-loop system are bounded, and there exists a positive constant Δ such that $\|\omega\| \le \Delta$ and $\|\hat{\omega}\| \le \Delta$. From Theorem 1, we claim that $\tilde{\bar{q}}$ and $\tilde{\omega}$ converge to zero in a finite time t_{f1}. When $t \ge t_{f1}$, V_1 and $\dot{V}_1$ become

$$V_1 = 2k_1(1-q_4) + \frac{1}{2}\hat{\omega}^TJ\hat{\omega} \tag{122}$$

$$\dot{V}_1 \le -k_2\hat{\omega}^T\text{sat}(\hat{\omega}) \le 0. \tag{123}$$

Furthermore, when $t \ge t_{f1}$, $\bar{q}$ and $\hat{\omega}$ are governed by the following dynamic equations

$$\begin{cases}\dot{\bar{q}} = M_1\hat{\omega}\\ J\dot{\hat{\omega}} = -\hat{\omega}^\times J\hat{\omega} - k_1\bar{q} - k_2\text{sat}(\hat{\omega}).\end{cases} \tag{124}$$

Now note that $\dot{V}_1 \equiv 0$ implies $\hat{\omega} \equiv 0$, which, in turn, implies $\bar{q} \equiv 0$ using (124). By LaSalle's invariant set theorem, the zero solution of system (124) is asymptotically stable, i.e., $\bar{q}$ and $\hat{\omega}$ converge to zero as $t \to \infty$. Since $\tilde{\omega}$ converges to zero in finite time, we conclude that ω converges to zero as $t \to \infty$.

References

[1] Jin E, Sun Z. Robust controllers design with finite time convergence for rigid spacecraft attitude tracking control. *Aerospace Science and Technology* 2008; 12 (4): 324-330.

[2] Zhu Z, Xia Y, Fu M. Attitude stabilization of rigid spacecraft with finite-time convergence. *International Journal of Robust and Nonlinear Control* 2011; 21 (6): 686-702.

[3] Li S, Wang Z, Fe S. Comments on the paper: Robust controllers design with finite time convergence for rigid spacecraft attitude tracking control. *Aerospace Science and Technology* 2011; 15 (3): 193-195.

[4] Lu K, Xia Y. Adaptive attitude tracking control for rigid spacecraft with finite-time convergence. *Automatica* 2013; 49 (12): 3591-3599.

[5] Zou AM, Kumar KD, Hou ZG, Liu X. Finite-time attitude tracking control for spacecraft using terminal sliding mode and Chebyshev neural network. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics* 2011; 41 (4): 950-963.

[6] Du H, Li S, Qian C. Finite-time attitude tracking control of spacecraft with application to attitude synchronization. *IEEE Transactions on Automatic Control* 2011; 56 (11): 2711-2717.

[7] Du H, Li S. Finite-time attitude stabilization for a spacecraft using homogeneous method. *AIAA Journal of Guidance, Control, and Dynamics* 2012; 35 (3): 740-748.

[8] Man Z, Paplinski AP, Wu HR. A robust MIMO terminal sliding mode control scheme for rigid robotic manipulators. *IEEE Transactions on Automatic Control* 1994; 39 (12): 2464-2469.

[9] Feng Y, Yu X, Man Z. Non-singular terminal sliding mode control of rigid manipulators. *Automatica* 2002; 38 (12): 2159-2167.

[10] Wang L, Chai T, Zhai L. Neural-network-based terminal sliding mode control of robotic manipulators including actuator dynamics. *IEEE Transactions Industrial Electronics* 2009; 56 (9): 3296-3304.

[11] Qian C, Lin W. A continuous feedback approach to global strong stabilization of nonlinear systems. *IEEE Transactions on Automatic Control* 2001; 46 (7): 1061-1079.

[12] Zou AM. Finite-time output feedback attitude tracking control for rigid spacecraft. *IEEE Transactions on Control Systems Technology* 2014; 22 (1): 338-345.

[13] Du H, Li S. Semi-global finite-time attitude stabilization by output feedback for a rigid spacecraft. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering* 2013; 227 (12): 1881-1891.

[14] Hughes PC. *Spacecraft Attitude Dynamics*. New York: Wiley, 1986.

[15] Sidi MJ. *Spacecraft Dynamics and Control: A Practical Engineering Approach*. Cambridge: Cambridge University Press, 1997.

[16] Hong Y, Wang J, Cheng D. Adaptive finite-time control of nonlinear systems with parametric uncertainty. *IEEE Transactions on Automatic Control* 2006; 51 (5): 858-862.

[17] Bhat S, Bernstein D. Continuous finite-time stabilization of the translational and rotational double integrators. *IEEE Transactions on Automatic Control* 1998; 43 (5): 678-682.

[18] Zou AM, de Ruiter A, Kumar KD. Distributed finite-time velocity-free attitude coordination control for spacecraft formations. *Automatica* 2016; 67: 46-53.

[19] Zou AM, Kumar KD, de Ruiter A. Robust attitude tracking control of spacecraft under control input magnitude and rate saturations. *International Journal of Robust and Nonlinear Control* 2016; 26 (4): 799-815.

[20] Bhat SP, Bernstein DS. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals, and Systems* 2005; (17): 101-127.

[21] Shen Y, Xia X. Semi-global finite-time observers for nonlinear systems. *Automatica* 2008; 44 (12): 3152-3156.

[22] Shen Y, Huang Y. Uniformly observable and globally Lipschitzian nonlinear systems admit global finite-time observers. *IEEE Transactions on Automatic Control* 2009; 54 (11): 2621-2625.

[23] Shuster MD. A survey of attitude representations. *Journal of Astronautical Sciences* 1993; 41 (4): 439-517.

[24] Wong H, de Queiroz MS, Kapila V. Adaptive tracking control using synthesized velocity from attitude measurements. *Automatica* 2001; 37 (6): 947-953.

[25] Lizarralde F, Wen JT. Attitude control without angular velocity measurement: a passivity approach. *IEEE Transactions on Automatic Control* 1996; 41 (3): 468-472.

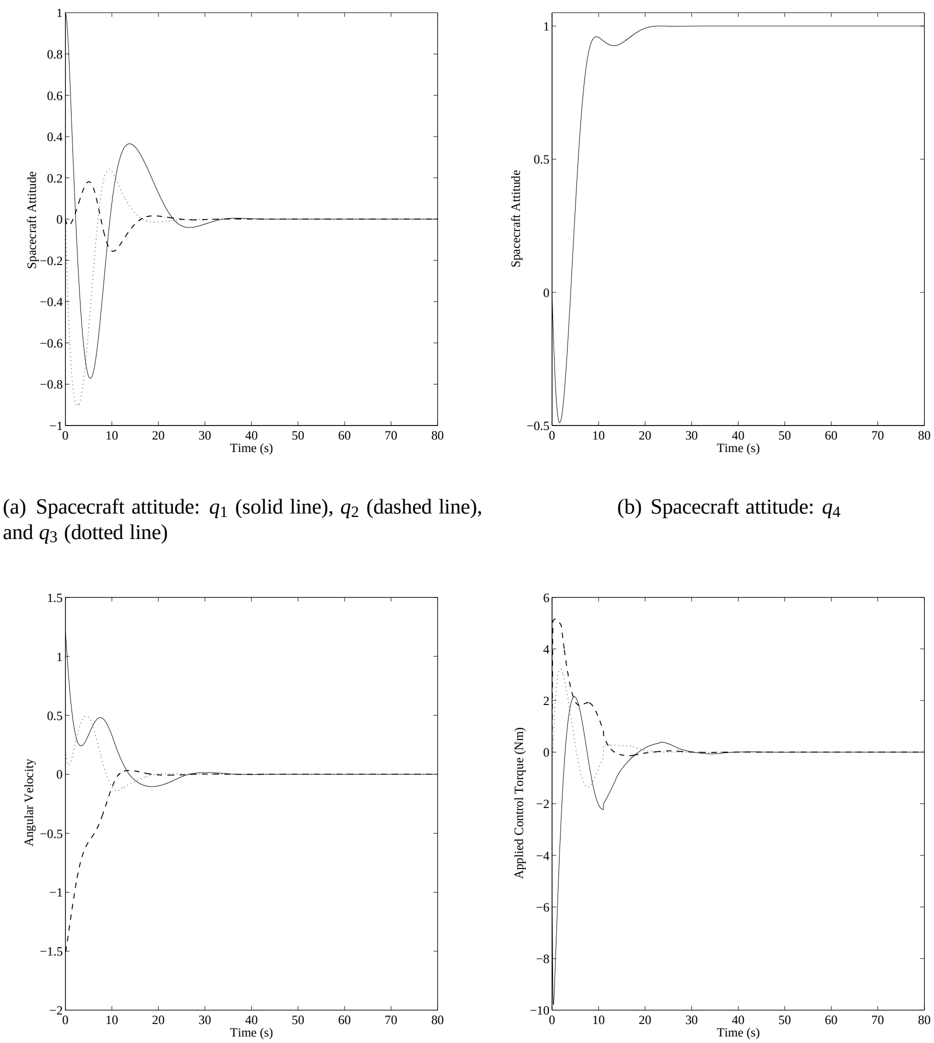
[26] Tsiotras P. Further passivity results for the attitude control problem. *IEEE Transactions on Automatic Control* 1998; 43 (11): 1597-1600.

[27] Caccavale F, Villani L. Output feedback control for attitude tracking. *Systems and Control Letters* 1999; 38 (2): 91-98.

[28] Tayebi A. Unit quaternion-based output feedback for the attitude tracking problem. *IEEE Transactions on Automatic Control* 2008; 53 (6): 1516-1520.

[29] Costic BT, Dawson DM, de Queiroz MS, Kapila V. Quaternion-based adaptive attitude tracking controller without velocity measurements. *Journal of Guidance, Control, and Dynamics* 2001; 24 (6): 1214-1222.

[30] Zou AM, Kumar KD, Hou ZG. Quaternion-based adaptive output feedback attitude control of spacecraft using Chebyshev neural networks. *IEEE Transactions on Neural Networks* 2010; 21 (9): 1457-1471.



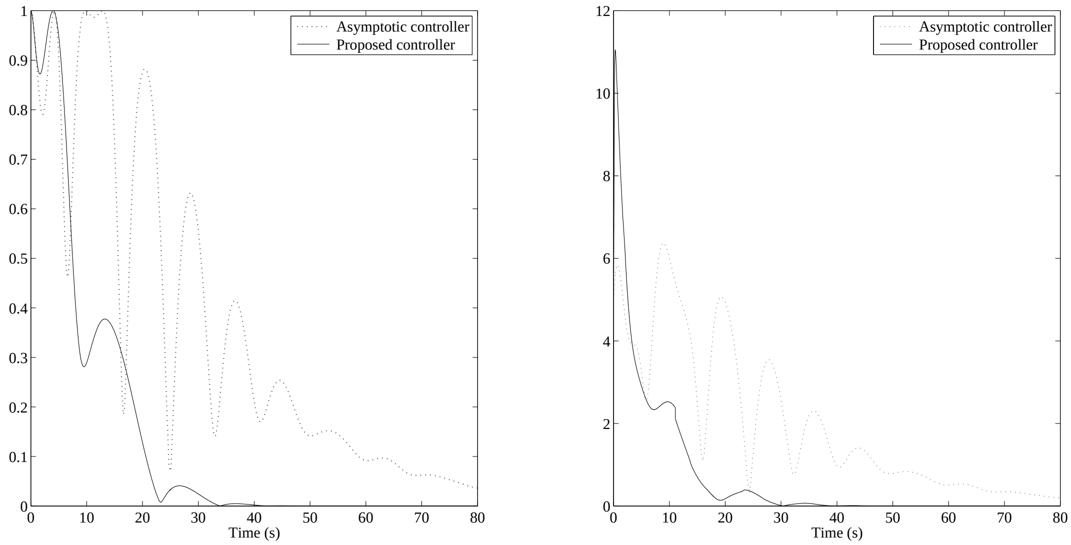
(a) Spacecraft attitude: q_1 (solid line), q_2 (dashed line), and q_3 (dotted line)

(b) Spacecraft attitude: q_4

(c) Angular velocity (rad/s): ω_1 (solid line), ω_2 (dashed line), and ω_3 (dotted line)

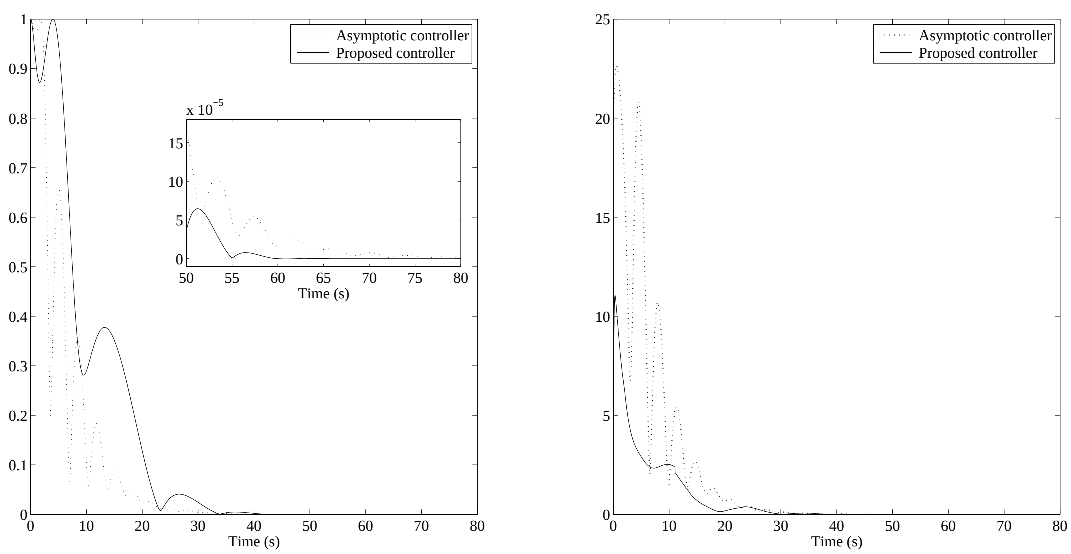
(d) Control torque: τ_1 (solid line), τ_2 (dashed line), and τ_3 (dotted line)

Figure 1. Effect of the proposed controller (47) on the attitude stabilization.



(a) Response of $\|\bar{q}\|$ (b) Response of $\|\tau\|$

Figure 2. Performance comparison between the proposed controller (47) and the controller in [28] without external disturbance ($a_1 = a_2 = 5$).



(a) Response of $\|\bar{q}\|$ (b) Response of $\|\tau\|$

Figure 3. Performance comparison between the proposed controller (47) and the controller in [28] without external disturbance ($a_1 = a_2 = 20$).

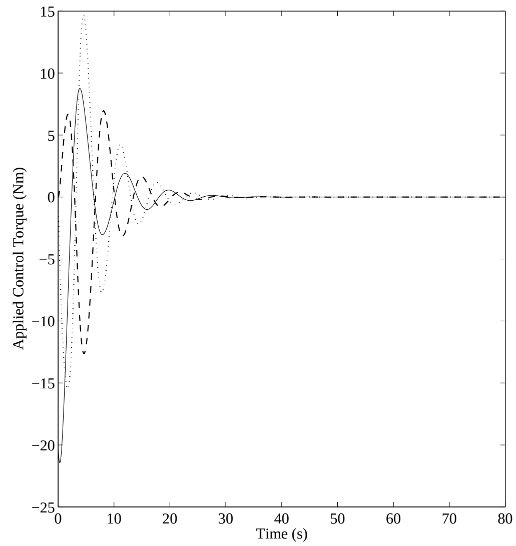
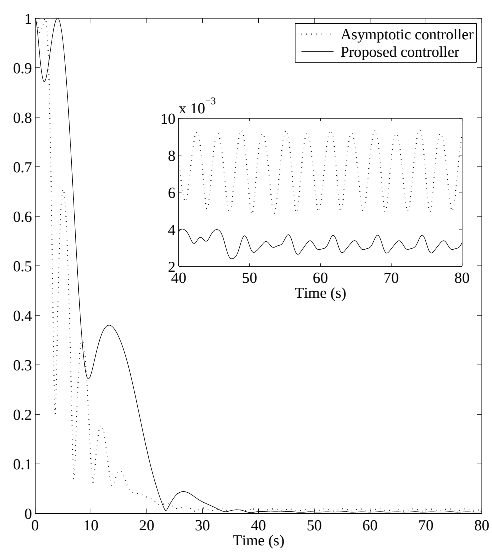
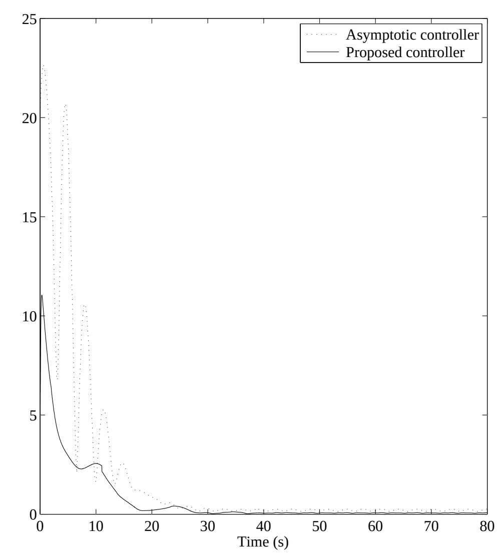


Figure 4. Control torque of the controller in [28] without external disturbance ($a_1 = a_2 = 20$): τ_1 (solid line), τ_2 (dashed line), and τ_3 (dotted line).

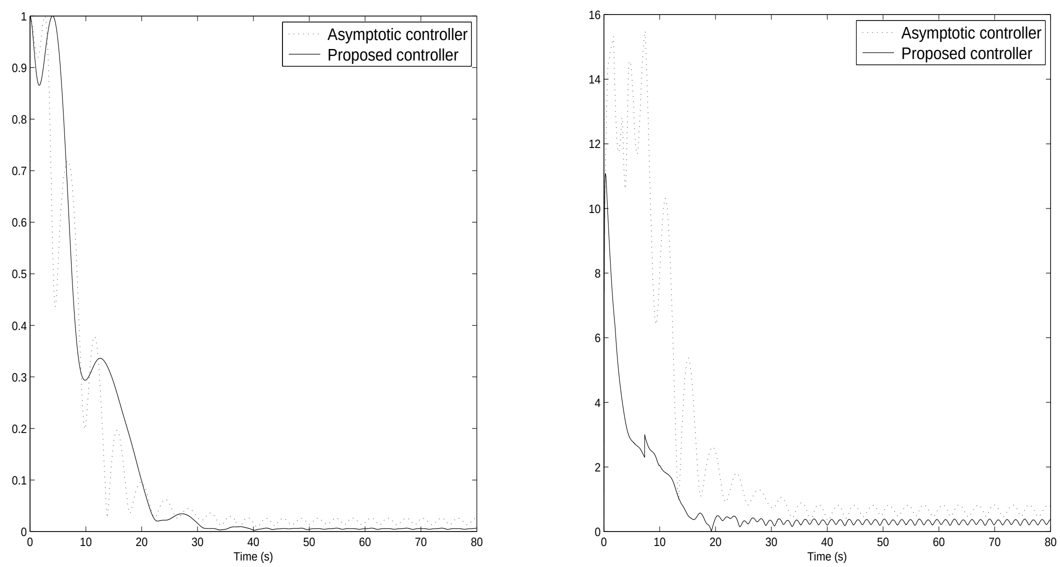


(a) Response of $\|\bar{q}\|$



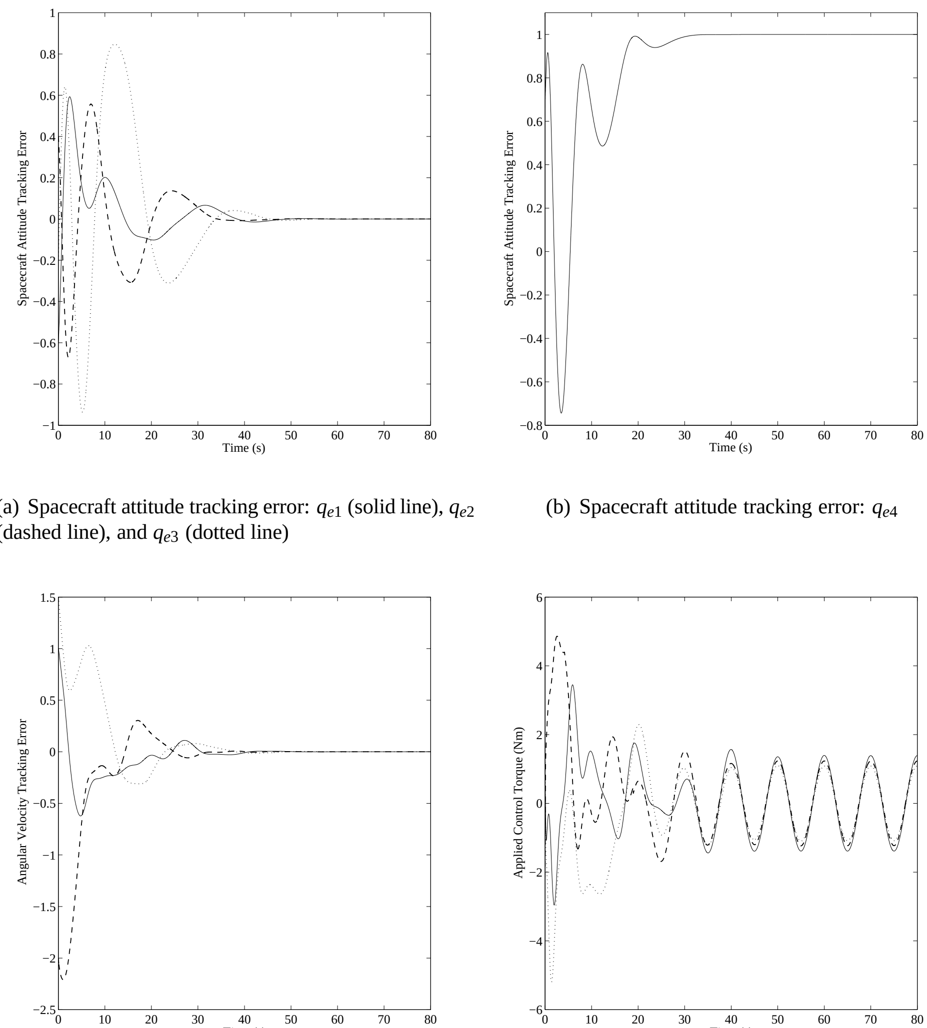
(b) Response of $\|\tau\|$

Figure 5. Performance comparison between the proposed controller (47) and the controller in [28] with external disturbance ($a_1 = a_2 = 20$).



(a) Response of $\|\bar{q}\|$ (b) Response of $\|\tau\|$

Figure 6. Performance comparison between the proposed controller (47) and the controller in [28] with measurement noise ($a_1 = a_2 = 20$).



(a) Spacecraft attitude tracking error: q_{e1} (solid line), q_{e2} (dashed line), and q_{e3} (dotted line)

(b) Spacecraft attitude tracking error: q_{e4}

(c) Angular velocity tracking error (rad/s): ω_{e1} (solid line), ω_{e2} (dashed line), and ω_{e3} (dotted line)

(d) Control torque: τ_1 (solid line), τ_2 (dashed line), and τ_3 (dotted line)

Figure 7. Effect of the proposed controller (90) on the attitude tracking.