Some Applications of Passivity-Based Control and Invariance Principles

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Abstract

This paper examines the control of a passive plant utilizing strictly passive feedback as motivated by the passivity theorem. A state representation of the plant is assumed with very few conditions imposed. No assumptions are made on the state representation of the feedback control law. However, some mild additional input-output properties of the feedback control law are assumed. Global stability of the closed-loop system and asymptotic convergence of a subset of the states is proven using invariance principles. The theoretical results in the paper are applied to a number of application examples, demonstrating that a much broader class of controllers can deliver closed-loop global stability and asymptotic convergence, unlike the previous works in the literature where the examples are taken from. In one of the examples, it is demonstrated that actuator saturation constraints can readily be handled using the theory presented in this paper.

1 Introduction

The passivity theorem [1] is a very important and powerful result in input-output theory. It has found many applications in numerous fields of control theory, particularly in the control of mechanical systems. Some examples include references [2] to [7], and there are numerous others. The passivity theorem states that a negative feedback connection between a passive and strictly passive system leads to input-output stability. As such, it is a powerful motivator for selecting control laws if a plant can be made passive. However, on its own, the passivity theorem only guarantees that the closed-loop output signals are contained in $\mathcal{L}_2$ provided that disturbance signals are contained in $\mathcal{L}_2$ (see the Appendix of this paper for a definition). Boundedness and asymptotic convergence of system states must be established using other means such as Lyapunov analysis [8]. In typical applications of the passivity theorem (such as in [2] to [7]), very specific forms of strictly passive control laws are selected to allow the Lyapunov-type stability analysis to be performed.

In [9], Willems introduced the concept of dissipativity as a generalization of passivity, and provided a condition under which an interconnection of dissipative systems is also dissipative. Then, a stability result for dissipative systems was presented, using Lyapunov analysis. It was further explained that to conclude asymptotic stability, other techniques such as invariance principles need to be used. The stability result in [9] is restricted to autonomous systems. References [10, 11], considered dissipative systems described by autonomous nonlinear affine-in-control differential equations. Conditions were provided for which global asymptotic stability could be concluded for these types of systems. The results rely on the additional concept of zero-state detectability, and make use invariance principles to obtain the final conclusions. Interconnections of autonomous affine-in-control passive systems form a special case of the results presented in [10, 11]. There has been significant work done in the area of passivity-based control over the past several decades. Excellent summaries may be found in references [12], [13] and [14], which provide several results governing the convergence of interconnected passive systems, typically making use of invariance principles. However, they generally rely on the assumption of underlying autonomous state-space models for each system. In [15], an extension of the Krasovskii-LaSalle theorem is made to asymptotically almost periodic (AAP) systems. This result is then used to prove asymptotic stability of passive affine-in-control AAP systems with a specific class of output feedback. However, the class of output feedback in [15] is restricted to memoryless proportional feedback.

Some further recent examples of passivity-based control are as follows. Reference [16] provides a stability analysis of radial power systems, using passivity theory and the concept of zero-state detectability together with invariance principles. The system under consideration in [16] is autonomous. In [17], passivity theory is used to develop an integral controller for a boost converter. The system is autonomous with a memoryless controller, and the convergence analysis relies on the concept of zero-state detectability together with invariance principles. In [18], the control of a Linear Time Invariant (LTI) plant with uncertain parameters is considered. By use of a loop transformation, a modified plant is obtained which is passive for all uncertain parameter combinations. A strictly passive LTI feedback control law is then designed for the modified plant, rendering the closed-loop system stable by virtue of the passivity theorem. By undoing the loop transformation, a robustly stabilizing LTI controller is obtained for the original uncertain plant. Reference [18] also presents a stability result for the feedback interconnection of an LTI system with a memoryless LTV system. To obtain conditions for stability, a loop transformation is again made, and the passivity theorem is applied requiring the modified LTV system to be passive, and the modified LTI system to be strictly passive. These stability conditions are then transformed into a sector condition on the original LTV system, and an LMI for the original LTI system. Reference [19] treats the tracking problem for Euler-Lagrange (EL) and Port Controlled Hamiltonian (PCH) systems. For Euler-Lagrange systems, a specific form of controller is obtained which renders passive the closed-loop map from an exogenous input to an output that is a function of the tracking error. A static damping term is then added to the controller which renders the closed-loop map output strictly passive, and hence stable. For PCH systems, a matching condition is presented such that the tracking error dynamics are also PCH, if the matching condition is satisfied by the controller. The asymptotic stability of the tracking error dynamics is then analysed using Lyapunov analysis together with LaSalle's invariance principle. Reference [20] considers passivity-based formation control of underwater vehicles. A specific form of controller is obtained, and asymptotic stability is proved using a nested Matrosov theorem [21], which is a persistency of excitation type theorem that can be used for stable nonautonomous systems (where conventional invariance principles do not apply). Reference [22] considers the output feedback tracking problem, for a single-input-single-output (SISO) LTI system in the presence of an uncertain nonlinear plant disturbance and output measurement noise, both with known bounds. The approach taken in [22] is to initially append a stable LTI filter to the plant input, and then design a specific form of feedback control law for the composite system rendering it passive. An outer-loop discontinuous switching control law is then designed to meet the output tracking objectives. The stability proof relies heavily on linear system theory.

In this paper, it is demonstrated how convergence analysis of output feedback for a class of passive systems may be performed by making use of invariance principles similar to [15]. Similar to [15], this paper considers passive asymptotically periodic nonlinear systems. Note that the assumption of asymptotic periodicity is slightly more restrictive than asymptotic almost periodicity. The reason for this additional restriction is for practical purposes; positive limit sets of asymptotically periodic systems are more readily determined than for asymptotically almost periodic systems. On the other hand, the more general nonlinear-in-control systems (rather than affine-in-control) are allowed. In addition, a broad class of output feedback controllers are allowed, which are solely characterized by some input-output properties. The theoretical results require a state-space model of the plant, while no state-space model is needed for the controller. This allows the convergence analysis to be performed with a significantly broader class of output feedback controller. The techniques used in obtaining the theoretical results in this paper are inspired by applications of an invariance principle for asymptotically autonomous systems, which are presented in LaSalle's seminal paper [23]. The usefulness of the theoretical results are demonstrated with a number of practical examples adapted from the literature. It is shown that using the theoretical results in this paper, a much broader class of controllers can deliver closed-loop global stability and asymptotic convergence, compared to the works in the literature where the examples are taken from.

The paper is structured as follows: Section 2 presents the main theoretical results. Section 3 presents the applications of the theoretical results to practical examples. Section 4 presents some concluding remarks. Finally, the Appendix to this paper contains some notation and preliminary mathematical results required in this paper.

2 Main Results

We shall consider plants of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{u},t\right), \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x},t), \end{aligned} \tag{1}$$

where the plant states are $\boldsymbol{x} \in \mathbb{R}^n$, the plant output is $\boldsymbol{y} \in \mathbb{R}^m$, the control input is $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{u},t\right)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ and uniformly bounded in $t \in \mathbb{R}^+$ on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m$. It is assumed that $\boldsymbol{h}(\boldsymbol{x},t)$ satisfies two properties: 1) $\boldsymbol{h}(\boldsymbol{x},t)$ is continuous in $\boldsymbol{x} \in \mathbb{R}^n$ uniformly in $t \in \mathbb{R}^+$, and uniformly continuous in $t \in \mathbb{R}^+$ uniformly in $\boldsymbol{x} \in \mathbb{R}^n$, 2) $\boldsymbol{h}(\mathbf{x}, t)$ is uniformly bounded in $t \in \mathbb{R}^+$ on any compact set $D \subset \mathbb{R}^n$,

We further assume that there exists a differentiable positive semi-definite function $V(\boldsymbol{x}, t) \geq 0$ with the property that for all $\gamma \geq 0$, the set $F_\gamma(t) \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : V(\boldsymbol{x}, t) \leq \gamma\}$ is uniformly bounded for all $t \in \mathbb{R}^+$. Furthermore, it is assumed that

$$\frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) + \frac{\partial V}{\partial t} \leq \boldsymbol{u}^T \boldsymbol{y} \tag{2}$$

for all $t \in \mathbb{R}^+$, all $\boldsymbol{x} \in \mathbb{R}^n$ and all $\boldsymbol{u} \in \mathbb{R}^m$. Note that this means that the map from $\boldsymbol{u}$ to $\boldsymbol{y}$ is passive [1, ch. 4].

Assumption 1

The feedback control laws considered in this paper are causal mappings $\mathcal{H}(\boldsymbol{v}) : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, satisfying

1. *$\mathcal{H}$ is input strictly passive.*
2. *$\boldsymbol{v} \in \mathcal{L}_\infty$ implies that $\boldsymbol{y} = \mathcal{H}(\boldsymbol{v}) \in \mathcal{L}_\infty$.*
3. *$\boldsymbol{v}(t) \to \mathbf{0}$ as $t \to \infty$ implies that $\boldsymbol{y}(t) \to \mathbf{0}$ as $t \to \infty$, where $\boldsymbol{y} = \mathcal{H}(\boldsymbol{v})$.*
4. *Continuity of $\boldsymbol{v}(t)$ implies continuity of $\mathcal{H}(\boldsymbol{v})$.*

By definition, Assumption 1.1 means that there exist $\delta > 0$ and $\beta \in \mathbb{R}$ such that for all $\boldsymbol{v} \in L_{2e}$

$$\int_0^T \mathcal{H}(\boldsymbol{v})(t)^T \boldsymbol{v}(t) dt \geq \delta \int_0^T \boldsymbol{v}(t)^T \boldsymbol{v}(t) dt + \beta, \;\; \forall T \geq 0. \tag{3}$$

Comments on the construction of $\boldsymbol{y} = \mathcal{H}(\boldsymbol{v})$ satisfying Assumption 1

All of the results in this paper require only that the feedback control laws satisfy the input-output properties given in Assumption 1. However, for the purpose of practical control implementation, we need to be able to physically realize such a control law in a form that can be implemented on a processor. Therefore, we shall now discuss a method by which feedback control laws satisfying Assumption 1 can be constructed. As we shall see, Assumption 1 is quite general, and allows for a very large class of control laws. What follows is but one way in which a suitable feedback control law may be constructed, and it is important to note that the class of controllers satisfying Assumption 1 is larger than those constructed in the following. Hence, to leave the results of this paper as general as possible, all stability and convergence results in this paper are based on the input-output properties of the feedback control law only.

As a starting point to constructing a feedback control law satisfying Assumption 1, consider the unforced system described by

$$\dot{\bar{\boldsymbol{z}}} = \boldsymbol{a}(\bar{\boldsymbol{z}}, t), \tag{4}$$

where $\bar{\boldsymbol{z}} \in \mathbb{R}^{n_c}$, $t \in \mathbb{R}^+$, and $\boldsymbol{a}(\bar{\boldsymbol{z}}, t)$ is locally Lipschitz continuous for all $\bar{\boldsymbol{z}} \in \mathbb{R}^{n_c}$, and piecewise continuous in $t \in \mathbb{R}^+$. Suppose that a known continuously differentiable Lyapunov-function $W(\bar{\boldsymbol{z}}, t)$ exists such that

$$k_1 \|\bar{\boldsymbol{z}}\|_p^q \leq W(\bar{\boldsymbol{z}}, t) \leq k_2 \|\bar{\boldsymbol{z}}\|_p^q, \tag{5}$$

$$\frac{\partial W}{\partial \bar{\boldsymbol{z}}} \boldsymbol{a}(\bar{\boldsymbol{z}}, t) + \frac{\partial W}{\partial t} \leq -k_3 \|\bar{\boldsymbol{z}}\|_p^q, \tag{6}$$

$$\left\| \frac{\partial W}{\partial \bar{\boldsymbol{z}}} \right\|_p \leq k_4 \|\bar{\boldsymbol{z}}\|_p^r, \tag{7}$$

where k_1, k_2, k_3, k_4, q and r are positive constants, with $q > r$, and conditions (5), (6) and (7) hold for all $(\bar{\boldsymbol{z}}, t) \in \mathbb{R}^{n_c} \times \mathbb{R}^+$. It is well known that under these conditions, the origin of (4) is globally exponentially stable [8]. It is also well-known that, given a globally exponentially stable system (4), a Lyapunov-function $W(\bar{\boldsymbol{z}}, t)$ can be found such that conditions (5), (6) and (7) hold with $q = 2$ and $r = 1$ [8]. We shall now show how a control law satisfying Assumption 1 may be constructed, using (4) together with (5) to (7).

Lemma 1

Consider the system (4), together with (5), (6) and (7). Suppose that the dimensionality of the input and output of mapping $\boldsymbol{y} = \mathcal{H}(\boldsymbol{v})$ is given as m_c. Let $\boldsymbol{B}(\boldsymbol{z}, t)$ be any continuous function on $\mathbb{R}^{n_c \times m_c} \times \mathbb{R}$, such that $\|\boldsymbol{B}(\boldsymbol{z}, t)\|_p \leq \bar{B}$ for all $(\boldsymbol{z}, t) \in \mathbb{R}^{n_c} \times \mathbb{R}$ and some finite $\bar{B} > 0$. Also, let $\boldsymbol{D}(\boldsymbol{z}, t)$ be any continuous function on

$\mathbb{R}^{m_c\times m_c}\times\mathbb{R}$ *such that* $\|\boldsymbol{D}(\boldsymbol{x},t)\|_p\le\bar{D}$, *and* $\sigma_{min}\left(\boldsymbol{D}(\boldsymbol{z},t)+\boldsymbol{D}^T(\boldsymbol{z},t)\right)\ge\delta_c$, *for all* $(\boldsymbol{z},t)\in\mathbb{R}^{n_c}\times\mathbb{R}$ *and some finite* $\bar{D}>0$ *and* $\delta_c>0$.

Then, the mapping from $\boldsymbol{y}=\mathcal{H}(\boldsymbol{v})$ *defined by*

$$\mathcal{H}:\ \begin{cases}\dot{\boldsymbol{z}} &= \boldsymbol{a}(\boldsymbol{z},t)+\boldsymbol{B}(\boldsymbol{z},t)\boldsymbol{v},\\ \boldsymbol{y} &= \boldsymbol{B}(\boldsymbol{z},t)^T\left(\partial W(\boldsymbol{z},t)/\partial\boldsymbol{z}\right)^T+\boldsymbol{D}(\boldsymbol{z},t)\boldsymbol{v},\end{cases}\tag{8}$$

satisfies Assumption 1, for any given $\boldsymbol{z}(0)$.

Proof

Causality of $\mathcal{H}$ is immediate, since it is realized by a set of state equations. To show condition 1 of Assumption 1, let us take the derivative of $W(\boldsymbol{z},t)$ along a trajectory of (8), for a given $\boldsymbol{z}(0)$. We obtain

$$\begin{aligned}\dot{W}(\boldsymbol{z},t) &= \frac{\partial W}{\partial\boldsymbol{z}}\boldsymbol{a}(\boldsymbol{z},t)+\frac{\partial W}{\partial t}+\frac{\partial W}{\partial\boldsymbol{z}}\boldsymbol{B}(\boldsymbol{z},t)\boldsymbol{v}(t),\\ &\le -k_3\|\boldsymbol{z}\|_p^q+\frac{\partial W}{\partial\boldsymbol{z}}\boldsymbol{B}(\boldsymbol{z},t)\boldsymbol{v}(t).\end{aligned}\tag{9}$$

By hypothesis, $\sigma_{min}\left(\boldsymbol{D}(\boldsymbol{z},t)+\boldsymbol{D}^T(\boldsymbol{z},t)\right)\ge\delta_c$. Therefore, making use of (8), inequality (9) can be upper-bounded by

$$\dot{W}(\boldsymbol{z},t)\le\boldsymbol{y}(t)^T\boldsymbol{v}(t)-\frac{\delta_c}{2}\boldsymbol{v}(t)^T\boldsymbol{v}(t).\tag{10}$$

Integrating both sides of (10) from $t=0$ to $t=T$ leads to

$$W(\boldsymbol{z}(T),T)-W(\boldsymbol{z}(0),0)\le\int_0^T\boldsymbol{y}(t)^T\boldsymbol{v}(t)dt-\frac{\delta_c}{2}\int_0^T\boldsymbol{v}^T\boldsymbol{v}(t)dt.\tag{11}$$

Recognizing that $W(\boldsymbol{z},t)\ge0$ (by (5)), inequality (11) can be rearranged to give

$$\int_0^T\boldsymbol{y}(t)^T\boldsymbol{v}(t)dt\ge\frac{\delta_c}{2}\int_0^T\boldsymbol{v}^T\boldsymbol{v}(t)dt+\beta,\tag{12}$$

where $\beta=-W(\boldsymbol{z}(0),0)$. Therefore, condition 1 of Assumption 1 holds.

To show condition 2 of Assumption 1, let $\boldsymbol{v}\in\mathcal{L}_\infty$. Then, $\|\boldsymbol{v}\|_p\le\bar{v}$ for all $t\ge0$ for some finite $\bar{v}>0$. By hypothesis, $\|\boldsymbol{B}(\boldsymbol{x},t)\|_p\le\bar{B}$. Therefore, making use of (7), inequality (9) can be upper-bounded by

$$\dot{W}(\boldsymbol{z},t)\le-\|\boldsymbol{z}\|_p^r\left(k_3\|\boldsymbol{z}\|_p^{q-r}-k_4\bar{B}\bar{v}\right).\tag{13}$$

Consider the set

$$F_\epsilon\triangleq\left\{\boldsymbol{z}\in\mathbb{R}^{n_c}:\|\boldsymbol{z}\|_p^{q-r}\le k_4\bar{B}\bar{v}/k_3+\epsilon\right\},\tag{14}$$

for some arbitrary $\epsilon>0$. Then,

$$\dot{W}(\boldsymbol{z},t)\le-\left(\frac{k_4\bar{B}\bar{v}}{k_3}+\epsilon\right)^{r/(q-r)}k_3\epsilon<0,\tag{15}$$

outside F_ϵ. Now, define the time-varying set

$$G_\epsilon(t)\triangleq\left\{\boldsymbol{z}\in\mathbb{R}^{n_c}:W(\boldsymbol{z},t)\le k_2\left(k_4\bar{B}\bar{v}/k_3+\epsilon\right)^{q/(q-r)}\right\}.\tag{16}$$

Then, by the right inequality in (5), $G_\epsilon(t)\supset F_\epsilon$ for all $t\ge0$, and consequently inequality (15) holds outside $G_\epsilon(t)$ also. Now, suppose that $\mathbf{z}(0)\in G_\epsilon(0)$. Then, by (15), it cannot leave $G_\epsilon(t)$ for all $t\ge0$. On the other hand, suppose that $\mathbf{z}(0)$ is outside $G_\epsilon(0)$. Then, by (15), $W(\boldsymbol{z}(t),t)\le W(\boldsymbol{z}(0),0)$ for all $t\ge0$. In both cases, $W(\boldsymbol{z}(t),t)$ is bounded, and by the left inequality in (5), we conclude that $\boldsymbol{z}\in\mathcal{L}_\infty$. Making use of the inequality (7) and the boundedness of $\boldsymbol{B}(\boldsymbol{z},t)$ and $\boldsymbol{D}(\boldsymbol{z},t)$, it follows that $\boldsymbol{y}\in\mathcal{L}_\infty$, which is condition 2 of Assumption 1.

To show condition 3 of Assumption 1, let $\boldsymbol{v}(t)\to\boldsymbol{0}$ as $t\to\infty$. Then, given any $\epsilon>0$, there exists a $T\ge0$ such that $\|\boldsymbol{v}(t)\|_p\le\epsilon$ for all $t\ge T$. Replacing $\bar{v}$ by ϵ in our proof of condition 2 of Assumption 1, we then find from (15) and (16) that

$$\dot{W}(\boldsymbol{z},t)\le-\left(\frac{k_4\bar{B}\epsilon}{k_3}+\epsilon\right)^{r/(q-r)}k_3\epsilon<0,\tag{17}$$

outside the set

$$\bar{G}_\epsilon(t) \triangleq \left\{ \boldsymbol{z} \in \mathbb{R}^{n_c} : W(\boldsymbol{z}, t) \le k_2 \left(k_4 \bar{B}\epsilon / k_3 + \epsilon \right)^{q/(q-r)} \right\}, \tag{18}$$

for all $t \ge T$. Therefore, there exists a $T' \ge T$ such that $\boldsymbol{z}(t) \in \bar{G}_\epsilon(t)$ for all $t \ge T'$. Now, by the left-hand inequality in (5), the set

$$H_\epsilon \triangleq \left\{ \boldsymbol{z} \in \mathbb{R}^{n_c} : \|\boldsymbol{z}\|_p^q \le \frac{k_2}{k_1} \left(k_4 \bar{B}\epsilon / k_3 + \epsilon \right)^{q/(q-r)} \right\}, \tag{19}$$

contains the set $\bar{G}_\epsilon(t)$ for all $t \ge 0$. Therefore, it must be that $\|\boldsymbol{z}(t)\|_p \le (k_2/k_1)^{1/q} \left(k_4 \bar{B}\epsilon / k_3 + \epsilon \right)^{1/(q-r)}$ for all $t \ge T'$. Finally, since $\epsilon > 0$ was arbitrary, it follows that $\boldsymbol{z}(t) \to \boldsymbol{0}$ as $t \to \infty$. Consequently, by (7) we find that $\partial W / \partial \boldsymbol{z} \to \boldsymbol{0}$. Finally, by the boundedness of $\boldsymbol{B}(\boldsymbol{z}, t)$ and $\boldsymbol{D}(\boldsymbol{z}, t)$, it follows that $\boldsymbol{y}(t) \to \boldsymbol{0}$ as $t \to \infty$.

Condition 4 of Assumption 1, follows from continuity of $\boldsymbol{z}(t)$ (solutions of (8) are absolutely continuous), and continuity of $\boldsymbol{B}(\boldsymbol{z}, t)$, $\partial W / \partial \boldsymbol{z}$ and $\boldsymbol{D}(\boldsymbol{z}, t)$. □

Let us now look at some special cases. First, let us restrict (4) to be linear, that is

$$\dot{\bar{\boldsymbol{z}}} = \boldsymbol{A}(t)\bar{\boldsymbol{z}}. \tag{20}$$

It is well-known [24] that (20) has globally exponentially stable origin if and only if for every symmetric, positive-definite, continuous and bounded matrix $\boldsymbol{Q}(t)$, there exists a symmetric and positive-definite continuously differentiable and bounded matrix $\boldsymbol{P}(t)$ that satisfies

$$\dot{\boldsymbol{P}}(t) + \boldsymbol{A}(t)\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{A}(t)^T = -\boldsymbol{Q}(t). \tag{21}$$

In this case, we can take $W(\bar{\mathbf{z}}, t) = (1/2)\bar{\boldsymbol{z}}^T \boldsymbol{P}(t)\bar{\boldsymbol{z}}$, and it is readily verified that together with (20), the conditions (5), (6) and (7) are satisfied with $p = q = 2$ and $r = 1$. In particular, we have

$$\frac{\partial W}{\partial \bar{\boldsymbol{z}}} = \bar{\boldsymbol{z}}^T \boldsymbol{P}(t).$$

Therefore, applying Lemma 1, we find that the linear mapping $\boldsymbol{y} = \mathcal{H}\boldsymbol{v}$ defined by

$$\mathcal{H}: \quad \left\{ \begin{array}{rcl} \dot{\boldsymbol{z}} & = & \boldsymbol{A}(t)\boldsymbol{z} + \boldsymbol{B}(t)\boldsymbol{v}, \\ \boldsymbol{y} & = & \boldsymbol{B}(t)^T \boldsymbol{P}(t)\boldsymbol{z} + \boldsymbol{D}(t)\boldsymbol{v}, \end{array} \right. \tag{22}$$

satisfies Assumption 1 for any bounded and continuous $\boldsymbol{B}(t)$ and $\boldsymbol{D}(t)$ provided $\sigma_{min}(\boldsymbol{D}(t) + \boldsymbol{D}(t)^T) \ge \delta_c$ for some $\delta_c > 0$, and $\boldsymbol{P}(t)$ satisfies (21) for some symmetric, positive-definite, continuous and bounded matrix $\boldsymbol{Q}(t)$. Note that (22) is very similar in form to the linear mappings provided in [7], which also satisfy Assumption 1.

Finally, let us restrict ourselves to static mappings. It is readily verified that the static mapping given by

$$\mathcal{H}: \quad \boldsymbol{y} = \boldsymbol{D}(t)\boldsymbol{v}, \tag{23}$$

for any bounded and continuous $\boldsymbol{D}(t)$ satisfying $\sigma_{min}(\boldsymbol{D}(t) + \boldsymbol{D}(t)^T) \ge \delta_c$ for some $\delta_c > 0$ also satisfies Assumption 1.

Remarks

1. We have now seen that it is straightforward to construct controllers satisfying Assumption 1. However, we shall not restrict ourselves to any particular realization of the mapping $\mathcal{H}$, so we will derive the main results in this paper based only on the input-output properties of the control law.

2. The feedback control laws given by (8) and (22) are general non-autonomous state-space systems. When applied to the plant (1), the passivity theorem can be used to conclude input-output stability of the closed-loop system. Boundedness and convergence analysis of the closed-loop states (or at least a subset of them) using conventional invariance principles requires additional restrictions on both the plant (1) and the controller (8), such as for example, asymptotic periodicity. The main contribution of this paper is to show how boundedness and convergence analysis may be performed on the closed-loop system using invariance principles without having to impose such additional restrictions on the controller (only the plant). In fact, only some input-output properties of the controller are required.

Main Results

We now state and prove the main results. In Theorem 1, we present a variant of the passivity theorem [1]. It is shown that under a suitable feedback control containing an input strictly passive map, the output of the considered system is convergent. This will form the basis of subsequent results.

Theorem 1

Let $\mathcal{H}$ satisfy conditions 1 and 2 of Assumption 1. Consider the system (1) with feedback control

$$\boldsymbol{u} = -\boldsymbol{K}^T(\boldsymbol{y},t)\mathcal{H}\left(\boldsymbol{K}(\boldsymbol{y},t)\boldsymbol{y}\right), \tag{24}$$

where $\boldsymbol{K}(\boldsymbol{y},t)$ continuous in $\boldsymbol{y}$ uniformly in t, and uniformly continuous in t uniformly in $\boldsymbol{y}$, uniformly bounded in $t \in \mathbb{R}^+$ on compact subsets of $\mathbb{R}^m$, and $\boldsymbol{K}(\boldsymbol{y}(t),t)\boldsymbol{y}(t) \to \boldsymbol{0}$ implies that $\boldsymbol{y}(t) \to \boldsymbol{0}$. Global existence of solutions of the closed-loop system is assumed.

Under these conditions, $\boldsymbol{x} \in \mathcal{L}_\infty$, $\boldsymbol{y} \in \mathcal{L}_\infty$ and $\boldsymbol{y}(t) \to \boldsymbol{0}$ as $t \to \infty$.

Proof

Let us take the derivative of $V(\boldsymbol{x},t)$ along a trajectory of the closed-loop system. We have

$$\dot{V} \leq \boldsymbol{y}^T\boldsymbol{u} = -\boldsymbol{y}^T\boldsymbol{K}^T(\boldsymbol{y},t)\mathcal{H}\left(\boldsymbol{K}(\boldsymbol{y},t)\boldsymbol{y}\right),$$

which leads to

$$\int_0^t \dot{V}d\tau \leq -\int_0^t \boldsymbol{y}^T\boldsymbol{K}^T(\boldsymbol{y},\tau)\mathcal{H}\left(\boldsymbol{K}(\boldsymbol{y},\tau)\boldsymbol{y}\right)(\tau)d\tau.$$

By the property (3), this leads to

$$V(\boldsymbol{x}(t),t) - V(\boldsymbol{x}(0),0) \leq -\delta\int_0^t \boldsymbol{y}^T\boldsymbol{K}^T(\boldsymbol{y}(\tau),\tau)\boldsymbol{K}(\boldsymbol{y}(\tau),\tau)\boldsymbol{y}(\tau)d\tau - \beta. \tag{25}$$

Rearranging (25), this shows that

$$V(\boldsymbol{x}(t),t) \leq V(\boldsymbol{x}(0),0) - \beta,$$

from which it can be concluded that $\boldsymbol{x} \in \mathcal{L}_\infty$. Since $\boldsymbol{h}(\boldsymbol{x},t)$ is uniformly bounded in $t \in \mathbb{R}^+$ on any compact set $D \subset \mathbb{R}^n$ we have $\boldsymbol{y} \in \mathcal{L}_\infty$. Since $\boldsymbol{K}(\boldsymbol{y}(t),t)$ is uniformly bounded in $t \in \mathbb{R}^+$ on compact subsets of $\mathbb{R}^m$, by Assumption 1.2 on $\mathcal{H}$, we find that $\boldsymbol{u} \in \mathcal{L}_\infty$. This implies that $(\boldsymbol{x}(t),\boldsymbol{u}(t)) \in D$ for some compact set $D \in \mathbb{R}^n \times \mathbb{R}^m$ for all $t \in \mathbb{R}^+$. By hypothesis then, $\boldsymbol{f}\left(\boldsymbol{x}(t),\boldsymbol{u}(t),t\right)$ is bounded for all $t \in \mathbb{R}^+$. Therefore, from (1), $\dot{\boldsymbol{x}}(t) \in \mathcal{L}_\infty$ and hence $\boldsymbol{x}(t)$ is uniformly continuous on $\mathbb{R}^+$. Since $\boldsymbol{x}(t)$ is bounded and uniformly continuous, we conclude from Proposition 1 (see the Appendix) that $\boldsymbol{y}(t)$ is uniformly continuous also. Likewise, we find that $\boldsymbol{K}(\boldsymbol{y}(t),t)$ is bounded and uniformly continuous and finally $\boldsymbol{K}(\boldsymbol{y}(t),t)\boldsymbol{y}(t)$ is bounded and uniformly continuous on $\mathbb{R}^+$. Rearranging (25) again, we have

$$\delta\int_0^t \boldsymbol{y}^T\boldsymbol{K}^T(\boldsymbol{y}(\tau),\tau)\boldsymbol{K}(\boldsymbol{y}(\tau),\tau)\boldsymbol{y}(\tau)d\tau \leq V(\boldsymbol{x}(0),0) - \beta.$$

Letting $t \to \infty$, this shows that $\boldsymbol{K}(\boldsymbol{y},t)\boldsymbol{y} \in \mathcal{L}_2$. From Barbalat's Lemma [8, p. 192], we can conclude that $\boldsymbol{K}(\boldsymbol{y}(t),t)\boldsymbol{y}(t) \to \boldsymbol{0}$ as $t \to \infty$, and hence by hypothesis $\boldsymbol{y}(t) \to \boldsymbol{0}$ as $t \to \infty$. This concludes the proof. □

Theorem 1 demonstrates that the output of the closed-loop system is convergent, however it makes no implications about the convergence of the system state. To make further conclusions about the asymptotic properties of the system state, additional restrictions are made on the system, which will allow the use of the invariance theory presented in the Appendix. Specifically, we now make the additional restriction that the system is asymptotically periodic, and that the output is time-independent. As shown in Section 3.1, such a system may arise in tracking problems for mechanical systems, and the usefulness of Theorem 2 is demonstrated by that example.

Theorem 2

Consider the system in (1), with $\boldsymbol{h}(\boldsymbol{x},t) = \boldsymbol{h}(\boldsymbol{x})$ and $\boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{u},t\right) = \tilde{\boldsymbol{f}}\left(\boldsymbol{x},\boldsymbol{u},\boldsymbol{g}(t)\right)$, where $\boldsymbol{g}(t) : \mathbb{R}^+ \to \mathbb{R}^p$ is a bounded continuous function on $t \in \mathbb{R}^+$, and $\tilde{\boldsymbol{f}}\left(\boldsymbol{x},\boldsymbol{u},\boldsymbol{g}\right)$ is continuous on $R^n \times \mathbb{R}^m \times \mathbb{R}^p$. Further assume that $\boldsymbol{g}(t) = \boldsymbol{g}_p(t) + \boldsymbol{g}_v(t)$ where $\boldsymbol{g}_p(t)$ is continuous and T-periodic and $\boldsymbol{g}_v(t) \to \boldsymbol{0}$ as $t \to \infty$. Let the conditions of Theorem 1 hold with the additional restriction that $\mathcal{H}$ satisfies conditions 3 and 4 of Assumption 1. Then, $\boldsymbol{x}(t)$ approaches the largest invariant set (in the sense of Corollary 1, located in the Appendix) of

$$\dot{\boldsymbol{x}} = \tilde{\boldsymbol{f}}\left(\boldsymbol{x},\boldsymbol{0},\boldsymbol{g}_p(t)\right), \tag{26}$$

contained in the set $E \stackrel{\triangle}{=} \{\boldsymbol{x} \in R^n : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}\}$.

Proof

From Theorem 1, $\boldsymbol{x} \in \mathcal{L}_\infty$ and $\boldsymbol{y}(t) \to \boldsymbol{0}$ as $t \to \infty$. Therefore, by Proposition 2 (see the Appendix), $\boldsymbol{x}(t)$ approaches the set E. Additionally, from the proof of Theorem 1, $\boldsymbol{K}(\boldsymbol{y}(t))\boldsymbol{y} \in L_2$, is continuous on $\mathbb{R}^+$ and $\boldsymbol{K}(\boldsymbol{y}(t), t)\boldsymbol{y}(t) \to \boldsymbol{0}$. Therefore, by assumption on $\mathcal{H}$, it must be that $\boldsymbol{u}(t)$ is continuous and $\boldsymbol{u}(t) \to \boldsymbol{0}$. Now, $\boldsymbol{x}(t)$ coincides with a solution $\boldsymbol{z}(t)$ of the system

$$\dot{\boldsymbol{z}}(t) = \tilde{\boldsymbol{f}}\left(\boldsymbol{z}, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)\right), \quad \boldsymbol{z}(0) = \boldsymbol{x}(0),$$

on $\mathbb{R}^+$, where

$$\bar{\boldsymbol{u}}(t) = \begin{cases} -\boldsymbol{K}^T(\boldsymbol{y}(t), t)\boldsymbol{w}(t), & t \geq 0, \\ -\boldsymbol{K}^T(\boldsymbol{y}(0), 0)\boldsymbol{w}(0), & t < 0, \end{cases}$$

$$\boldsymbol{w}(t) = \mathcal{H}\left(\boldsymbol{K}(\boldsymbol{y}, t)\boldsymbol{y}\right)(t)$$

$$\bar{\boldsymbol{g}}(t) = \begin{cases} \boldsymbol{g}(t), & t \geq 0, \\ \boldsymbol{g}(0), & t < 0, \end{cases}$$

and $\boldsymbol{y}(t) = \boldsymbol{h}\left(\boldsymbol{x}(t)\right)$.

Since $\tilde{\boldsymbol{f}}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, it is uniformly continuous on compact subsets $D \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$. From the proof of Theorem 1, $\boldsymbol{u}(t)$ is bounded on $\mathbb{R}^+$, $\boldsymbol{g}(t)$ is bounded on $\mathbb{R}^+$ by hypothesis. Therefore $(\bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)) \in D_1 \subset \mathbb{R}^m \times \mathbb{R}^p$ for all $t \in \mathbb{R}$. Now, consider any compact subset $D_2 \subset \mathbb{R}^n$. Then, $\tilde{\boldsymbol{f}}$ is uniformly continuous on $D_2 \times D_1$. Therefore, for any $\epsilon > 0$, there exists a $\zeta > 0$ such that for any $\boldsymbol{z} \in D_2$,

$$\|\left(\bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)\right) - \left(\boldsymbol{0}, \boldsymbol{g}_p(t)\right)\|_p = \|\left(\boldsymbol{z}, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)\right) - \left(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{g}_p(t)\right)\|_p < \zeta$$

implies that

$$\|\tilde{\boldsymbol{f}}\left(\boldsymbol{z}, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)\right) - \tilde{\boldsymbol{f}}\left(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{g}_p(t)\right)\|_p < \epsilon.$$

Since $\bar{\boldsymbol{u}}(t) \to \boldsymbol{0}$ and $\boldsymbol{g}_v(t) \to \boldsymbol{0}$, there exists a $t^* \geq 0$ such that $\forall t \geq t^*$, $\|\left(\bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)\right) - \left(\boldsymbol{0}, \boldsymbol{g}_p(t)\right)\|_p < \zeta$. This shows that $\tilde{\boldsymbol{f}}\left(\boldsymbol{z}, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)\right) - \tilde{\boldsymbol{f}}\left(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{g}_p(t)\right) \to \boldsymbol{0}$ as $t \to \infty$, uniformly in $\boldsymbol{z}$ on compact subsets of $\mathbb{R}^n$. We also note that continuity of $\bar{\boldsymbol{u}}(t)$, $\bar{\boldsymbol{g}}(t)$ and $\boldsymbol{g}_p(t)$ imply continuity of $\tilde{\boldsymbol{f}}\left(\boldsymbol{z}, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}(t)\right)$ and $\tilde{\boldsymbol{f}}\left(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{g}_p(t)\right)$ on $\mathbb{R}^n \times \mathbb{R}$.

It has been shown that all conditions of Corollary 1 are satisfied. Therefore, the positive limit set of $\boldsymbol{z}(t)$ and therefore of $\boldsymbol{x}(t)$ is invariant under (26). Since $\boldsymbol{x}(t)$ is bounded, it must approach its positive limit set [8, p. 114]. The conclusion now follows. □

Theorem 2 considered the case where the system itself is asymptotically periodic. However, the situation may arise where the system contains a non-vanishing time-varying non-periodic part, but this part appears as a product with a subset of the states. As shown in Section 3.2, such a system may arise in tracking problems for spacecraft attitude, and the usefulness of Theorem 3 is demonstrated in that example.

Theorem 3

Let the conditions of Theorem 1 hold with the additional restriction that $\mathcal{H}$ satisfies conditions 3 and 4 of Assumption 1, and assume that the state-space can be partitioned into $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1^T & \boldsymbol{x}_2^T \end{bmatrix}^T$ (where $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$) such that

$$\boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, t\right) = \begin{bmatrix} \boldsymbol{f}_1\left(\boldsymbol{a}_1(\boldsymbol{x}_1, t), \boldsymbol{x}_2, \boldsymbol{u}, \boldsymbol{g}_1(t)\right) \\ \boldsymbol{f}_2\left(\boldsymbol{a}_2(\boldsymbol{x}_1, t), \boldsymbol{x}_2, \boldsymbol{u}, \boldsymbol{g}_2(t)\right) \end{bmatrix}, \tag{27}$$

where $\boldsymbol{g}_1(t)$ and $\boldsymbol{g}_2(t)$ are bounded continuous functions satisfying $\boldsymbol{g}_1(t) = \boldsymbol{g}_{1p}(t) + \boldsymbol{g}_{1v}(t)$, $\boldsymbol{g}_2(t) = \boldsymbol{g}_{2p}(t) + \boldsymbol{g}_{2v}(t)$, with $\boldsymbol{g}_{1p}(t)$ and $\boldsymbol{g}_{2p}(t)$ continuous and T-periodic (with the same period T) and $\boldsymbol{g}_{1v}(t), \boldsymbol{g}_{2v}(t) \to \boldsymbol{0}$ as $t \to \infty$, and $\boldsymbol{a}_1(\boldsymbol{x}_1, t)$ and $\boldsymbol{a}_2(\boldsymbol{x}_1, t)$ are continuous, with the properties that

1. *Given any compact subset $D \in \mathbb{R}^{n_1}$, there exist $\bar{a}_1, \bar{a}_2 > 0$ such that $\boldsymbol{x}_1 \in D$ implies that $\|\boldsymbol{a}_1(\boldsymbol{x}_1, t)\|_p \leq \bar{a}_1$ and $\|\boldsymbol{a}_2(\boldsymbol{x}_1, t)\|_p \leq \bar{a}_2$, for all $t \in \mathbb{R}^+$.*

2. *$\boldsymbol{x}_1(t) \to \boldsymbol{0}$ as $t \to \infty$ implies that $\boldsymbol{a}_1(\boldsymbol{x}_1(t), t) \to \boldsymbol{0}$ and $\boldsymbol{a}_2(\boldsymbol{x}_1(t), t) \to \boldsymbol{0}$ as $t \to \infty$.*

Suppose that $\boldsymbol{y}(t) \to \boldsymbol{0}$ implies that $\boldsymbol{x}_1(t) \to \boldsymbol{0}$. Then $\boldsymbol{x}_1(t) \to \boldsymbol{0}$ as $t \to \infty$ and $\boldsymbol{x}_2(t)$ approaches the largest invariant set of

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1\left(\boldsymbol{0}, \boldsymbol{x}_2, \boldsymbol{0}, \boldsymbol{g}_{1p}(t)\right), \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2\left(\boldsymbol{0}, \boldsymbol{x}_2, \boldsymbol{0}, \boldsymbol{g}_{2p}(t)\right), \end{aligned} \tag{28}$$

contained in the set $E \overset{\Delta}{=} \{(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}^n : \boldsymbol{x}_1 = \boldsymbol{0}\}$.

Proof

From Theorem 1, $\boldsymbol{y}(t) \to \boldsymbol{0}$. Therefore, by hypothesis, $\boldsymbol{x}_1(t) \to \boldsymbol{0}$. From the proof of Theorem 2, $\boldsymbol{u}(t)$ is continuous and $\boldsymbol{u}(t) \to \boldsymbol{0}$. Now, $(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t))$ coincides with a solution $(\boldsymbol{z}_1(t), \boldsymbol{z}_2(t))$ of the system

$$\begin{array}{rcl} \dot{\boldsymbol{z}}_1 &=& \boldsymbol{f}_1\left(\bar{\boldsymbol{w}}_1(t), \boldsymbol{z}_2, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}_1(t)\right), \quad \boldsymbol{z}_1(0) = \boldsymbol{x}_1(0) \\ \dot{\boldsymbol{z}}_2 &=& \boldsymbol{f}_2\left(\bar{\boldsymbol{w}}_2(t), \boldsymbol{z}_2, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}_2(t)\right), \quad \boldsymbol{z}_2(0) = \boldsymbol{x}_2(0), \end{array} \tag{29}$$

on $\mathbb{R}^+$, where $\bar{\boldsymbol{u}}(t)$, $\bar{\boldsymbol{g}}_1(t)$ and $\bar{\boldsymbol{g}}_2(t)$ are defined in the same manner as in the proof of Theorem 2, and similarly,

$$\bar{\boldsymbol{w}}_i(t) = \begin{cases} \boldsymbol{a}_i(\boldsymbol{x}_1(t), t), & t \geq 0, \\ \boldsymbol{a}_i(\boldsymbol{x}_1(0), 0), & t < 0, \end{cases} \quad i = 1, 2.$$

Now, the same arguments as in the proof of Theorem 2 can be used to conclude that

$$\boldsymbol{f}_1\left(\bar{\boldsymbol{w}}_1(t), \boldsymbol{z}_2, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}_1(t)\right) - \boldsymbol{f}_1\left(\boldsymbol{0}, \boldsymbol{z}_2, \boldsymbol{0}, \boldsymbol{g}_{1p}(t)\right) \to \boldsymbol{0},$$

and

$$\boldsymbol{f}_2\left(\bar{\boldsymbol{w}}_2(t), \boldsymbol{z}_2, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}_2(t)\right) - \boldsymbol{f}_2\left(\boldsymbol{0}, \boldsymbol{z}_2, \boldsymbol{0}, \boldsymbol{g}_{2p}(t)\right) \to \boldsymbol{0},$$

uniformly in $\boldsymbol{z}_2$ on compact subsets of $\mathbb{R}^{n_2}$ as $t \to \infty$, and that $\boldsymbol{f}_1\left(\bar{\boldsymbol{w}}_1(t), \boldsymbol{z}_2, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}_1(t)\right)$, $\boldsymbol{f}_2\left(\bar{\boldsymbol{w}}_2(t), \boldsymbol{z}_2, \bar{\boldsymbol{u}}(t), \bar{\boldsymbol{g}}_2(t)\right)$, $\boldsymbol{f}_1\left(\boldsymbol{0}, \boldsymbol{z}_2, \boldsymbol{0}, \boldsymbol{g}_{1p}(t)\right)$ and $\boldsymbol{f}_2\left(\boldsymbol{0}, \boldsymbol{z}_2, \boldsymbol{0}, \boldsymbol{g}_{2p}(t)\right)$ are continuous on $\mathbb{R}^{n_2} \times \mathbb{R}$. Finally, from Theorem 1, $(\boldsymbol{z}_1(t), \boldsymbol{z}_2(t)) \equiv (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)) \in \mathcal{L}_\infty$. Therefore, from Corollary 1, the positive limit set of $(\boldsymbol{z}_1(t), \boldsymbol{z}_2(t))$ and therefore $(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t))$ is invariant under

$$\begin{array}{rcl} \dot{\boldsymbol{z}}_1 &=& \boldsymbol{f}_1\left(\boldsymbol{0}, \boldsymbol{z}_2, \boldsymbol{0}, \boldsymbol{g}_{1p}(t)\right), \\ \dot{\boldsymbol{z}}_2 &=& \boldsymbol{f}_2\left(\boldsymbol{0}, \boldsymbol{z}_2, \boldsymbol{0}, \boldsymbol{g}_{2p}(t)\right). \end{array}$$

Additionally, $(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t))$ must approach its positive limit set [8, p. 114]. Since $\boldsymbol{x}_1(t) \to \boldsymbol{0}$, the conclusion follows. □

3 Application Examples

We now demonstrate the usefulness of the main results through three specific applications. The are applied to various problems found in the literature. It is shown here how the results in this paper may be used to establish global asymptotic convergence for a much broader class of controllers.

3.1 Gain-Scheduled Tracking Control of a Rigid Manipulator

The dynamics of a rigid manipulator are described by [2]

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{30}$$

where $\boldsymbol{q}$ is a vector containing the joint angles, $\boldsymbol{\tau}$ contains the corresponding joint torques, $\boldsymbol{M}(\boldsymbol{q})$ is a uniformly bounded positive definite mass matrix, $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is contains centrifugal and coriolis terms such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is skew-symmetric, and $\boldsymbol{g}(\boldsymbol{q})$ is derived from a potential function $P(\boldsymbol{q})$ such that $\boldsymbol{g}(\boldsymbol{q}) = \partial P / \partial \boldsymbol{q}$.

Let $\boldsymbol{q}_r(t)$ be a desired joint trajectory, satisfying the following assumption:

Assumption 2

$\boldsymbol{q}_r(t)$, $\dot{\boldsymbol{q}}_r(t)$, $\ddot{\boldsymbol{q}}_r(t)$ *are continuous and bounded. In addition,* $\ddot{\boldsymbol{q}}_r(t)$ *is asymptotically periodic.*

Consider the control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{K}\left(\boldsymbol{q} - \boldsymbol{q}_r\right) + \boldsymbol{u}, \tag{31}$$

where $\boldsymbol{K} = \boldsymbol{K}^T > \boldsymbol{0}$ is constant, and $\boldsymbol{u}$ is an auxiliary input. Defining the tracking error to be

$$\tilde{\boldsymbol{q}} \overset{\Delta}{=} \boldsymbol{q} - \boldsymbol{q}_r, \tag{32}$$

and substituting (31) into (30) leads to the error dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\tilde{\boldsymbol{q}}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} = -\boldsymbol{K}\tilde{\boldsymbol{q}} + \boldsymbol{u}. \tag{33}$$

As in [5, 6], we choose a gain scheduled control law of the form

$$\boldsymbol{u} = -\sum_{i=1}^{N} s_i\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) \mathcal{H}_i\left(s_i\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) \dot{\tilde{\boldsymbol{q}}}\right), \tag{34}$$

where N is the number of controllers contained in the scheduling law, each $\mathcal{H}_i$ satisfies Assumption 1 for $i = 1, ..., N$, and $s_i\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right)$ are scheduling signals satisfying

$$\sum_{i=1}^{N} s_i^2\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) \geq \alpha > 0. \tag{35}$$

Additionally, each $s_i\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right)$ $(i = 1, ..., N)$ is required to be continuous in all of its arguments and uniformly bounded in time for bounded $(\boldsymbol{q}, \dot{\boldsymbol{q}})$. It can then readily be shown using the techniques in [5, 6] that the mapping defined by

$$\mathcal{H}(\boldsymbol{v}) = \sum_{i=1}^{N} s_i\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) \mathcal{H}_i\left(s_i\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) \boldsymbol{v}\right), \tag{36}$$

also satisfies all conditions in Assumption 1 for bounded and continuous $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$.

We can now write the system and error dynamics in the form of (1) by setting

$$\boldsymbol{x} = \left[\begin{array}{cccc} \dot{\tilde{\boldsymbol{q}}}^T & \tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T & \boldsymbol{q}^T \end{array}\right]^T, \tag{37}$$

$$\tilde{\boldsymbol{f}}\left(\boldsymbol{x}, \boldsymbol{u}, \ddot{\boldsymbol{q}}_r(t)\right) = \left[\begin{array}{c} \boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} - \boldsymbol{K}\tilde{\boldsymbol{q}} + \boldsymbol{u}\right) \\ \dot{\tilde{\boldsymbol{q}}} \\ \ddot{\boldsymbol{q}}_r(t) + \boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} - \boldsymbol{K}\tilde{\boldsymbol{q}} + \boldsymbol{u}\right) \\ \dot{\boldsymbol{q}} \end{array}\right]. \tag{38}$$

Due to (32), we consider only trajectories of (37) and (38) with initial conditions satisfying (32) at $t = 0$. We take as output $\boldsymbol{y} = \dot{\tilde{\boldsymbol{q}}}$. Now, we choose $V(\boldsymbol{x}, t) = \frac{1}{2}\dot{\tilde{\boldsymbol{q}}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T \boldsymbol{K}\tilde{\boldsymbol{q}}$. We readily find that $(\partial V/\partial \boldsymbol{x})\, \tilde{\boldsymbol{f}} + \partial V/\partial t = \boldsymbol{y}^T \boldsymbol{u}$. Choosing the feedback control law (34), and applying the first part of the proof of Theorem 1, we find that $\tilde{\boldsymbol{q}}(t)$ and $\dot{\tilde{\boldsymbol{q}}}(t)$ are bounded. Using Assumption 2, we find that $\boldsymbol{q}(t)$ and $\dot{\boldsymbol{q}}(t)$ are bounded also and the result of Theorem 1 follows. Since $\ddot{\boldsymbol{q}}_r(t)$ is asymptotically periodic, the following result is readily obtained by application of Theorem 2.

Theorem 4

Let Assumption 2 be satisfied. Consider the rigid manipulator with dynamics given by (30) with control law (31) and (34). Then, $\tilde{\boldsymbol{q}}(t) \in \mathcal{L}_\infty$, $\dot{\tilde{\boldsymbol{q}}}(t) \in \mathcal{L}_\infty$, *and* $\tilde{\boldsymbol{q}}(t), \dot{\tilde{\boldsymbol{q}}}(t) \rightarrow \boldsymbol{0}$ *as* $t \rightarrow \infty$.

Remark

It has been shown in [5, 6] that in the known parameter case, improved tracking performance for a manipulator may be obtained by scheduling input strictly passive controllers, each optimized at various points in the configuration space. While [5, 6] both treat flexible manipulators, for brevity we have restricted ourselves to rigid manipulators. Of references [5] and [6], only [5] performs a convergence analysis of the tracking error. Reference [6] restricts itself to the demonstration of input-output stability. In [5], in order to prove convergence of the tracking error to zero, constant set-points are assumed, and the gain-scheduled controllers are restricted to be linear time-invariant with given state-space representations. Using the results presented in Section 2 of this paper, the class of trajectories to be tracked in this example has been significantly broadened from constant set-points to trajectories with asymptotically periodic second derivative. Additionally, only some input-output properties of the scheduled controllers are required to establish convergence of the tracking error to zero, significantly broadening the class of scheduled controllers as well.

3.2 Spacecraft Attitude Tracking Control

In the next two sections, we consider spacecraft attitude tracking control. There are many results on spacecraft attitude tracking in the literature, some classic examples being [3] and [4]. References [3] and [4] consider adaptive spacecraft attitude tracking, where the spacecraft inertia matrix is unkown. The control laws in [3] and [4] are developed on the basis of the so-called filtered error. This leads to a feedforward term in the control law which contains both the desired body-rate as well as the attitude error. In this section, we consider non-adaptive spacecraft attitude tracking, which allows us to use a simpler feedforward term that does not contain the attitude error. The adaptive case will be dealt with in Section 3.3. In [3] and [4], the feedback component of the control law is a constant proportional output feedback law. Using the results presented in Section 2, we obtain a much broader class of feedback controllers requiring only some input-output properties in order to guarantee convergence of the attitude and body-rate errors to zero.

In body coordinates, the attitude dynamics of a spacecraft are [25, pp.157, 237]

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{\tau}_c, \tag{39}$$

where $\boldsymbol{I}$ is the spacecraft inertia matrix, $\boldsymbol{\omega}$ is the angular velocity, and $\boldsymbol{\tau}_c$ is a control torque. The definition of $\boldsymbol{\omega}^{\times}$ may be found in the Appendix.

Let the desired attitude have corresponding angular velocity $\boldsymbol{\omega}_r(t)$ (expressed in true body coordinates). We restrict ourselves to bounded and differentiable $\boldsymbol{\omega}_r(t)$. We shall describe the attitude error by the quaternion $(\boldsymbol{q}, q_4)$ [25, p. 31], representing the rotation from desired to true attitude. The angular velocity error is defined as

$$\tilde{\boldsymbol{\omega}} \stackrel{\Delta}{=} \boldsymbol{\omega} - \boldsymbol{\omega}_r(t). \tag{40}$$

Accordingly, the attitude error kinematics obey [25, p. 31]

$$\dot{\boldsymbol{q}} = \frac{1}{2}\left[\boldsymbol{q}^{\times} + q_4\boldsymbol{1}\right]\tilde{\boldsymbol{\omega}}, \;\; \dot{q}_4 = -\frac{1}{2}\boldsymbol{q}^T\tilde{\boldsymbol{\omega}}. \tag{41}$$

Let the control law be

$$\boldsymbol{\tau}_c = \boldsymbol{I}\dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}_r^{\times}\boldsymbol{I}\boldsymbol{\omega} - k\boldsymbol{q} + \boldsymbol{u}, \;\; k > 0, \tag{42}$$

where $\boldsymbol{u}$ is an auxiliary input. Substituting (42) into (39), we obtain

$$\boldsymbol{I}\dot{\tilde{\boldsymbol{\omega}}} = -\tilde{\boldsymbol{\omega}}^{\times}\boldsymbol{I}\boldsymbol{\omega} - k\boldsymbol{q} + \boldsymbol{u}. \tag{43}$$

Combining this with (41), we can write the system of equations in the form (1) with (27) by setting

$$\boldsymbol{x}_1 = \tilde{\boldsymbol{\omega}}, \;\; \boldsymbol{x}_2 = \left[\begin{array}{cc} \boldsymbol{q}^T & q_4 \end{array}\right]^T,$$

$$\boldsymbol{f}_1\left(\boldsymbol{a}_1(\boldsymbol{x}_1, t), \boldsymbol{x}_2, \boldsymbol{u}\right) = \boldsymbol{a}_1(\boldsymbol{x}_1, t) - k\boldsymbol{I}^{-1}\bar{\boldsymbol{x}}_2 + \boldsymbol{I}^{-1}\boldsymbol{u}, \;\; \boldsymbol{x}_2 = \left[\begin{array}{c} \bar{\boldsymbol{x}}_2 \\ x_{2,4} \end{array}\right] \tag{44}$$

$$\boldsymbol{f}_2\left(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}\right) = \frac{1}{2}\left[\begin{array}{c} \bar{\boldsymbol{x}}_2^{\times} + x_{2,4}\boldsymbol{1} \\ -\bar{\boldsymbol{x}}_2^T \end{array}\right]\boldsymbol{x}_1, \tag{45}$$

and

$$\boldsymbol{a}_1(\boldsymbol{x}_1, t) = \boldsymbol{I}^{-1}\left(\boldsymbol{I}(\boldsymbol{\omega}^r(t) + \boldsymbol{x}_1)\right)^{\times}\boldsymbol{x}_1. \tag{46}$$

Since $\boldsymbol{\omega}^r(t)$ is bounded, it is clear that $\boldsymbol{a}_1(\boldsymbol{x}_1, t)$ satisfies the conditions required in Theorem 3.

Now, as output we choose $\boldsymbol{y} = \boldsymbol{x}_1$, which can be done in this case, since $\boldsymbol{x}_1$ has the same dimension as $\boldsymbol{u}$. Setting $V(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2}\boldsymbol{x}_1^T\boldsymbol{I}\boldsymbol{x}_1 + k\left[\bar{\boldsymbol{x}}_2^T\bar{\boldsymbol{x}}_2 + (x_{2,4} - 1)^2\right]$, we readily find that $(\partial V/\partial \boldsymbol{x}_1)\,\boldsymbol{f}_1 + (\partial V/\partial \boldsymbol{x}_2)\,\boldsymbol{f}_2 = \boldsymbol{y}^T\boldsymbol{u}$.

Choosing the feedback control law $\boldsymbol{u} = -\mathcal{H}(\boldsymbol{y})$, where $\mathcal{H}$ satisfies Assumption 1, the first part of the proof of Theorem 1 shows that $\boldsymbol{y}(t) = \boldsymbol{x}_1(t) = \tilde{\boldsymbol{\omega}}(t)$ is bounded. It is clear now that all conditions of Theorem 1 are satisfied, and we can obtain the following result by application of Theorem 3.

Theorem 5

Consider the spacecraft attitude tracking problem specified by (39), (40), (56) and (42), with feedback control law $\boldsymbol{u} = -\mathcal{H}(\tilde{\boldsymbol{\omega}})$*, where* $\mathcal{H}$ *satisfies Assumption 1. Then,* $(\tilde{\boldsymbol{\omega}}(t), \boldsymbol{q}(t)) \rightarrow \boldsymbol{0}$ *as* $t \rightarrow \infty$*.*

3.3 Adaptive Spacecraft Attitude Tracking Control

Let us now consider an adaptive version of the spacecraft attitude tracking control problem, where the spacecraft inertia matrix $\boldsymbol{I}$ is not known a-priori. The control law is a modification and generalization of that presented in [4]. As stated in Section 3.2, the feedback component of the control law in [3] and [4] is a constant output feedback law. Using the results presented in Section 2, we obtain a much broader class of feedback controllers requiring only some input-output properties in order to guarantee convergence of the attitude and body-rate errors to zero.

Before continuing, the following Lemma will be required.

Lemma 2 [30].
Consider the filtered error given by $\boldsymbol{r}(t) \triangleq \tilde{\boldsymbol{\omega}}(t) + \lambda \boldsymbol{q}(t)$*, where* $\lambda > 0$ *and* $\tilde{\boldsymbol{\omega}}$ *and* $\boldsymbol{q}$ *satisfy (56). Then,* $\boldsymbol{r}(t) \to \boldsymbol{0}$ *as* $t \to \infty$ *if and only if* $\tilde{\boldsymbol{\omega}}(t) \to \boldsymbol{0}$ *and* $\boldsymbol{q}(t) \to \boldsymbol{0}$ *as* $t \to \infty$*.*

We consider again the problem described in Section 3.2 described by equations (39), (40) and (56). We make the following restriction on $\boldsymbol{\omega}_r(t)$:

Assumption 2
The reference angular velocity $\boldsymbol{\omega}_r(t)$ *and its derivative* $\dot{\boldsymbol{\omega}}_r(t)$ *are bounded and continuous.*

Since we do not know the spacecraft inertia matrix, we modify the feedforward control in (42) to

$$\boldsymbol{\tau}_c = \hat{\boldsymbol{I}} \dot{\boldsymbol{\omega}}_d + \boldsymbol{\omega}_d^{\times} \hat{\boldsymbol{I}} \boldsymbol{\omega} + \boldsymbol{u}, \tag{47}$$

where $\boldsymbol{u}$ is an auxiliary control input, $\hat{\boldsymbol{I}}$ is an estimate of the inertia matrix and

$$\boldsymbol{\omega}_d = \boldsymbol{\omega}_r - \lambda \boldsymbol{q}, \;\; \lambda > 0. \tag{48}$$

Substituting (47) into (39), and defining the inertia estimate error to be $\tilde{\boldsymbol{I}} = \hat{\boldsymbol{I}} - \boldsymbol{I}$ we obtain

$$\boldsymbol{I} \dot{\boldsymbol{r}} = -\boldsymbol{r}^{\times} \boldsymbol{I} \boldsymbol{\omega} + \boldsymbol{W} \left(\dot{\boldsymbol{\omega}}_d, \boldsymbol{\omega}_d, \boldsymbol{\omega} \right) \tilde{\boldsymbol{\theta}} + \boldsymbol{u}, \tag{49}$$

where

$$\boldsymbol{r} = \tilde{\boldsymbol{\omega}} + \lambda \boldsymbol{q}, \tag{50}$$

is the commonly known filtered error, $\tilde{\boldsymbol{\theta}} = \text{vec}\{\tilde{\boldsymbol{I}}\}$, is a vector of the independent entries of $\tilde{\boldsymbol{I}}$ ($\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are similarly defined), and $\boldsymbol{W} \left(\dot{\boldsymbol{\omega}}_d, \boldsymbol{\omega}_d, \boldsymbol{\omega} \right)$ is a regressor matrix satisfying

$$\boldsymbol{W} \left(\dot{\boldsymbol{\omega}}_d, \boldsymbol{\omega}_d, \boldsymbol{\omega} \right) \tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{I}} \dot{\boldsymbol{\omega}}_d + \boldsymbol{\omega}_d^{\times} \tilde{\boldsymbol{I}} \boldsymbol{\omega}. \tag{51}$$

Now, suppose that it is known that the true entries of the spacecraft inertia matrix lie in the set

$$F \triangleq \left\{ \boldsymbol{\upsilon} \in R^6 : \|\boldsymbol{\upsilon} - \boldsymbol{\theta}^*\|_2 \leq p \right\},$$

for some nominal inertia values $\boldsymbol{\theta}^*$ and tolerance $p > 0$. Let us define the function

$$d(\boldsymbol{\upsilon}) = \frac{\|\boldsymbol{\upsilon} - \boldsymbol{\theta}^*\|_2^2 - p^2}{\epsilon_\upsilon p^2}, \;\; \epsilon_\upsilon > 0, \tag{52}$$

and define the projection operator [31]

$$\text{Proj}\left(\boldsymbol{\upsilon}, \boldsymbol{y}\right) = \begin{cases} \boldsymbol{y} - \frac{d(\boldsymbol{\upsilon}) \nabla d(\boldsymbol{\upsilon}) \nabla d(\boldsymbol{\upsilon})^T \boldsymbol{y}}{\|\nabla d(\boldsymbol{\upsilon})\|_2^2}, & d(\boldsymbol{\upsilon}) > 0 \text{ and } \nabla d(\boldsymbol{\upsilon})^T \boldsymbol{y} > 0, \\ \boldsymbol{y}, & \text{otherwise.} \end{cases} \tag{53}$$

We select the adaptation law

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma} \text{Proj}\left(\hat{\boldsymbol{\theta}}, -\boldsymbol{W} \left(\dot{\boldsymbol{\omega}}_d, \boldsymbol{\omega}_d, \boldsymbol{\omega} \right)^T \boldsymbol{r} \right), \;\; \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > \boldsymbol{0}. \tag{54}$$

Combining equations (49) and (54), and noting that $\dot{\tilde{\boldsymbol{\theta}}} = \dot{\hat{\boldsymbol{\theta}}}$, we can write the system of equations in the form of (1) by setting

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r} \\ \tilde{\boldsymbol{\theta}} \end{bmatrix},$$

$$
\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},(t)) = \begin{bmatrix} \boldsymbol{I}^{-1}\left[-\boldsymbol{r}^{\times}\boldsymbol{I}\boldsymbol{\omega}(t) + \boldsymbol{W}\left(\dot{\boldsymbol{\omega}}_d(t),\boldsymbol{\omega}_d(t),\boldsymbol{\omega}(t)\right)\tilde{\boldsymbol{\theta}} + \boldsymbol{u}\right] \\ \boldsymbol{\Gamma}\mathrm{Proj}\left(\hat{\boldsymbol{\theta}}(t), -\boldsymbol{W}\left(\dot{\boldsymbol{\omega}}_d(t),\boldsymbol{\omega}_d(t),\boldsymbol{\omega}(t)\right)^T\boldsymbol{r}\right) \end{bmatrix}.
$$

As output, we choose $\boldsymbol{y} = \boldsymbol{r}$. Now, it can be readily shown that

$$
(\boldsymbol{\upsilon} - \boldsymbol{\theta})^T \left(\mathrm{Proj}\left(\boldsymbol{\upsilon},\boldsymbol{y}\right) - \boldsymbol{y}\right) \le \mathbf{0},
$$

for any $\boldsymbol{\upsilon} \in \mathbb{R}^6$, $\boldsymbol{y} \in \mathbb{R}^6$ provided $\boldsymbol{\theta} \in F$ [31]. Therefore, choosing $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{r}^T\boldsymbol{I}\boldsymbol{r} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}$, it can readily be shown that $\dot{V} \le \boldsymbol{y}^T\boldsymbol{u}$.

By Assumption 2, $\boldsymbol{\omega}_r(t)$ and $\dot{\boldsymbol{\omega}}_r(t)$ are bounded and continuous. Choosing the feedback control law as in Theorem 2, the first part of the proof of Theorem 2 shows that $\boldsymbol{r}(t)$ is bounded and continuous. Since $\|\boldsymbol{q}(t)\|_2 \le 1$ (a property of the quaternion) it must be that $\tilde{\boldsymbol{\omega}}(t)$ is bounded and continuous also. Making use of Assumption 2, (41) and (48) it can be concluded that $\boldsymbol{\omega}(t)$, $\boldsymbol{\omega}_d(t)$, $\dot{\boldsymbol{\omega}}_d(t)$ are bounded and continuous also. It is a property of the projection operator (53) that if $\hat{\theta}(0) \in \bar{F}$, where $\bar{F} \triangleq \left\{\boldsymbol{\upsilon} \in \mathbb{R}^6 : \|\boldsymbol{\upsilon} - \boldsymbol{\theta}^*\|_2 \le \sqrt{1+\epsilon_\upsilon}p\right\}$, then with the adaptation law (54), $\hat{\boldsymbol{\theta}}(t) \in \bar{F}$ for all $t \in \mathbb{R}^+$ [31] and $\hat{\boldsymbol{\theta}}(t)$ is continuous. It is clear now that all conditions of Theorem 2 are satisfied, and we can obtain the following result.

Theorem 6

Let Assumption 2 hold. Consider the spacecraft attitude tracking problem specified by (39), (40), (56) and (47), with feedback control law $\mathbf{u} = -\mathbf{K}^T(\mathbf{r}(t),t)\mathcal{H}\left(\mathbf{K}(\mathbf{r}(t),t)\mathbf{r}(t)\right)$*, where* $\mathcal{H}$ *satisfies Assumption 1 and* $\mathbf{K}(\mathbf{r}(t),t)$ *satisfies the assumptions stated in Theorem 2. Then,* $\tilde{\boldsymbol{\omega}}(t) \in \mathcal{L}_\infty$*,* $\mathbf{q} \in \mathcal{L}_\infty$ *and* $(\tilde{\boldsymbol{\omega}}(t),\mathbf{q}(t)) \to \mathbf{0}$ *as* $t \to \infty$*.*

Proof

As already demonstrated, from Theorem 2, we see that $\boldsymbol{r}(t) \in \mathcal{L}_\infty$. Since $\|\boldsymbol{q}(t)\|_2 \le 1$ (a property of the quaternion) it must be that $\tilde{\omega}(t) \in \mathcal{L}_\infty$. Additionally, from Theorem 2 we find that $\boldsymbol{y}(t) = \boldsymbol{r}(t) \to \mathbf{0}$. Therefore, the final conclusion follows from Lemma 2. □

Remark

The fact that $\boldsymbol{q}(t) \to \mathbf{0}$ could have been obtained directly from an application of the passivity theorem [1, ch. 4] in the case $\boldsymbol{K} = \mathbf{1}$, and the fact that $\boldsymbol{r}(t) \in \mathcal{L}_2$ implies that $\boldsymbol{q}(t) \to \mathbf{0}$ [4]. However, boundedness of all signals and convergence of $\tilde{\boldsymbol{\omega}}(t)$ require an alternative approach. In other works (such as for example [4, 2, 3], typically a very specific form of feedback law $\mathcal{H}$ is chosen, and a Lyapunov approach is used to establish the additional properties. The choice of the feedback control law using the results in this paper is much more general.

3.4 Hybrid Magnetic and Mechanical Attitude Control with Actuator Constraints

A hybrid magnetic and mechanical control scheme for spacecraft attitude regulation was presented in [7]. In this section, it is shown how actuator saturation constraints (which are not considered in [7]) may be incorporated into a broad class of control laws, while guaranteeing convergence to zero of the attitude error.

The attitude dynamics considered in [7] are expressed in spacecraft body coordinates and are given by (39) where $\boldsymbol{\tau}_c = \boldsymbol{\tau}_w + \boldsymbol{\tau}_m$ and $\boldsymbol{\tau}_w$ and $\boldsymbol{\tau}_m$ are control torques applied by mechanical and magnetic actuators respectively. The magnetic torque satisfies

$$
\boldsymbol{\tau}_m = \boldsymbol{b}^{\times T}\boldsymbol{m}, \tag{55}
$$

where $\boldsymbol{b}$ is the local Earth magnetic field vector, and $\boldsymbol{m}$ is the magnetic torquer dipole moment. The spacecraft inertial attitude is represented by the quaternion $(\boldsymbol{q}, q_4)$, and the attitude kinematics are given by [25, p. 31]

$$
\dot{\boldsymbol{q}} = \frac{1}{2}\left[\boldsymbol{q}^{\times} + q_4\mathbf{1}\right]\boldsymbol{\omega}, \;\; \dot{q}_4 = -\frac{1}{2}\boldsymbol{q}^T\boldsymbol{\omega}. \tag{56}
$$

The control input is designed as

$$
\boldsymbol{\tau}_w + \boldsymbol{\tau}_m = -k\boldsymbol{q} + \boldsymbol{u}, \;\; k > 0, \tag{57}
$$

where $\boldsymbol{u}$ is an auxiliary input. Substituting (57) into (39) gives the augmented plant

$$
\boldsymbol{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} - k\boldsymbol{q} + \boldsymbol{u}. \tag{58}
$$

Combining this with (56), we can write the system of equations in the form (1) with (27) by setting

$$\boldsymbol{x}_1 = \boldsymbol{\omega}, \;\; \boldsymbol{x}_2 = \begin{bmatrix} \boldsymbol{q}^T & q_4 \end{bmatrix}^T,$$

$$\boldsymbol{f}_1\left(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}\right) = -\boldsymbol{I}^{-1}\boldsymbol{x}_1^{\times}\boldsymbol{I}\boldsymbol{x}_1 - k\boldsymbol{I}^{-1}\bar{\boldsymbol{x}}_2 + \boldsymbol{I}^{-1}\boldsymbol{u}, \;\; \boldsymbol{x}_2 = \begin{bmatrix} \bar{\boldsymbol{x}}_2 \\ x_{2,4} \end{bmatrix} \tag{59}$$

$$\boldsymbol{f}_2\left(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}\right) = \frac{1}{2}\begin{bmatrix} \bar{\boldsymbol{x}}_2^{\times} + x_{2,4}\boldsymbol{1} \\ -\bar{\boldsymbol{x}}_2^T \end{bmatrix}\boldsymbol{x}_1, \tag{60}$$

As output, we choose $\boldsymbol{y} = \boldsymbol{x}_1$, which can be done in this case, since $\boldsymbol{x}_1$ has the same dimension as $\boldsymbol{u}$. Setting $V(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2}\boldsymbol{x}_1^T\boldsymbol{I}\boldsymbol{x}_1 + k\left[\bar{\boldsymbol{x}}_2^T\bar{\boldsymbol{x}}_2 + (x_{2,4}-1)^2\right]$, we readily find that $(\partial V/\partial \boldsymbol{x}_1)\,\boldsymbol{f}_1 + (\partial V/\partial \boldsymbol{x}_2)\,\boldsymbol{f}_2 = \boldsymbol{y}^T\boldsymbol{u}$. In [7], the feedback control law is chosen as

$$\boldsymbol{u} = -\delta\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}^T\boldsymbol{\omega} - \hat{\boldsymbol{b}}^{\times T}\mathcal{H}\left(\hat{\boldsymbol{b}}^{\times}\boldsymbol{\omega}\right), \tag{61}$$

where $\mathcal{H}$ is a linear map satisfying Assumption 1, and $\delta > 0$ is associated with $\mathcal{H}$ in the sense of (3), and $\hat{\boldsymbol{b}} = \boldsymbol{b}/\|\boldsymbol{b}\|_2$. The first term is applied by mechanical actuators, while the second term is applied by the magnetic torquers. It is shown in [7] that (61) satisfies the all conditions in Assumption 1 with input $\boldsymbol{\omega}$. Combining (57) and (61), the mechanical and magnetic control inputs in [7] are given by

$$\boldsymbol{\tau}_w = \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}^T\left[-k\boldsymbol{q} - \delta\boldsymbol{\omega}\right], \tag{62}$$

and

$$\boldsymbol{m} = \|\boldsymbol{b}\|_2^{-1}\left[-k\hat{\boldsymbol{b}}^{\times}\boldsymbol{q} - \mathcal{H}\left(\hat{\boldsymbol{b}}^{\times}\boldsymbol{\omega}\right)\right]. \tag{63}$$

Note that (63) leads to

$$\boldsymbol{\tau}_m = \hat{\boldsymbol{b}}^{\times T}\left[-k\hat{\boldsymbol{b}}^{\times}\boldsymbol{q} - \mathcal{H}\left(\hat{\boldsymbol{b}}^{\times}\boldsymbol{\omega}\right)\right].$$

We now modify the control law to account for magnetic torquer saturation. Let $m_{\max}$ represent the maximum capability of the magnetic torquer such that $|\boldsymbol{m}_i| \le m_{\max}$ for $i = x, y, z$. Clearly, if $\|\boldsymbol{m}\|_2 \le m_{\max}$, then the magnetic torquer control commands will never exceed their capacity. Now, it is well known that the quaternion satisfies $\boldsymbol{q} \le 1$ [25, ch. 3]. Therefore, $\|\boldsymbol{b}\|_2^{-1}\|k\hat{\boldsymbol{b}}^{\times}\boldsymbol{q}\|_2 \le k/b_{\min}$ (since $\|\hat{\boldsymbol{b}}^{\times}\|_2 = 1$) where $b_{\min} = \inf_{t\in\mathbb{R}^+}\|\boldsymbol{b}(t)\|_2$. Clearly, we must choose $0 < k < m_{\max}b_{\min}$. Now, for the chosen $\mathcal{H}$ in [7], a finite $\gamma > 0$ can be found such that $\|\boldsymbol{w}(t)\|_2 \le \gamma\sup_{t\in\mathbb{R}^+}\|\boldsymbol{v}(t)\|_2$ for all $t \in \mathbb{R}^+$, where $\boldsymbol{w} = \mathcal{H}(\boldsymbol{v})$. Let us now choose

$$K(\boldsymbol{v}) = \begin{cases} 1, & \|\boldsymbol{v}\|_2 \le \frac{m_{\max}b_{\min}-k}{\gamma}, \\ \frac{m_{\max}b_{\min}-k}{\gamma\|\boldsymbol{v}\|_2}, & \|\boldsymbol{v}\|_2 > \frac{m_{\max}b_{\min}-k}{\gamma}. \end{cases} \tag{64}$$

Then, $\sup_{t\in\mathbb{R}^+}\|K(\boldsymbol{v}(t))\boldsymbol{v}(t)\|_2 \le \frac{m_{\max}b_{\min}-k}{\gamma}$ and consequently $\sup_{t\in\mathbb{R}^+}\|\bar{\boldsymbol{w}}(t)\|_2 \le m_{\max}b_{\min} - k$, where $\bar{\boldsymbol{w}} = \mathcal{H}\left(K(\boldsymbol{v})\boldsymbol{v}\right)$. It is also readily shown that $K(\boldsymbol{v})$ is continuous in $\boldsymbol{v}$, and $K(\boldsymbol{v}(t))\boldsymbol{v}(t) \to \boldsymbol{0}$ if and only if $\boldsymbol{v}(t) \to \boldsymbol{0}$. Let us now modify the control law (61) to

$$\boldsymbol{u} = -\delta K(\boldsymbol{\omega})^2\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}^T\boldsymbol{\omega} - K(\boldsymbol{\omega})\hat{\boldsymbol{b}}^{\times T}\mathcal{H}\left(K(\boldsymbol{\omega})\hat{\boldsymbol{b}}^{\times}\boldsymbol{\omega}\right), \tag{65}$$

so that the corresponding mechanical and magnetic control inputs (including the proportional control in (61)) are given by

$$\boldsymbol{\tau}_w = \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}^T\left[-k\boldsymbol{q} - \delta K(\boldsymbol{\omega})^2\boldsymbol{\omega}\right], \tag{66}$$

and

$$\boldsymbol{m} = \|\boldsymbol{b}\|_2^{-1}\left[-k\hat{\boldsymbol{b}}^{\times}\boldsymbol{q} - K(\boldsymbol{\omega})\mathcal{H}\left(K(\boldsymbol{\omega})\hat{\boldsymbol{b}}^{\times}\boldsymbol{\omega}\right)\right]. \tag{67}$$

From (67), it is readily concluded that $\|\boldsymbol{m}\|_2 \le m_{\max}$. Global asymptotic stability of the closed-loop system can now be concluded by application of Theorem 3. In summary, we obtain the following result.

Theorem 7

Let the magnetic torquer capability be limited by $|\boldsymbol{m}_i| \le m_{\max}$ for $i = x, y, z$. Consider the spacecraft attitude regulation problem specified by (39), (55), (56) and (57), with feedback control law (65) with (64), where $0 < k < m_{\max}b_{\min}$. Then, the magnetic torquer commands never exceed the torquer capabilities, and $(\boldsymbol{\omega}(t), \boldsymbol{q}(t)) \to \boldsymbol{0}$ as $t \to \infty$.

4 Conclusion

Motivated by the passivity theorem, we have examined the control of a passive plant with strictly passive feedback control laws. We have assumed a very general state representation of the plant with very few conditions imposed. We have assumed some mild additional input-output properties for the feedback control law, however no state representation assumptions were made for the feedback control law. Making use of invariance theory for asymptotically periodic systems, we have proven global stability of the closed-loop system and asymptotic convergence of a subset of the states. The theoretical results have been applied to gain scheduled tracking of a rigid manipulator, non-adaptive and adaptive spacecraft attitude tracking, and hybrid magnetic/mechanical spacecraft attitude regulation. These examples have demonstrated that a much broader class of controllers can deliver closed-loop global stability and asymptotic convergence, unlike the previous works in the literature where the examples were taken from. In the hybrid magnetic/mechanical spacecraft attitude regulation example, it was demonstrated that actuator saturation constraints are readily treated using the theory presented in this paper.

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Appendix

In this Appendix, we outline some notation as well as some mathematical results needed in the main part of the paper.

The identity matrix is denoted by **1** so as to distinguish it from the spacecraft inertia matrix. The cross-product operator matrix associated with a vector $\boldsymbol{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T \in \mathbb{R}^3$ is given by

$$\boldsymbol{a}^\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

We take the norm $\|\boldsymbol{x}\|_p$ of a vector $\boldsymbol{x}$ to mean the usual Hölder p-norm for $p = 1, 2, ...$ The norm $\|\boldsymbol{X}\|_p$ of a matrix $\boldsymbol{X}$ is taken to mean the usual Hölder induced p-norm. The minimum singular value of a matrix $\boldsymbol{X}$ is denoted by $\sigma_{min}(\boldsymbol{X})$. The norm of a vector signal $\boldsymbol{u}(t)$ is taken to be $\|\boldsymbol{u}\|_{L2} = \sqrt{\int_0^\infty \boldsymbol{u}^T(t)\boldsymbol{u}(t)dt}$. The $\mathcal{L}_2$-space is defined as $\mathcal{L}_2 \triangleq \{\boldsymbol{u} : \|\boldsymbol{u}\|_{L2} < \infty\}$. The truncation of a signal is defined as

$$\boldsymbol{u}_T(t) = \begin{cases} \boldsymbol{u}(t), & t \leq T, \\ \boldsymbol{0}, & t > T. \end{cases}$$

The $\mathcal{L}_{2e}$-space is defined as $\mathcal{L}_{2e} \triangleq \{\boldsymbol{u} : \boldsymbol{u}_T(\in L_2, 0 \leq T < \infty\}$.

Finally, the $\mathcal{L}_\infty$ space is defined as $\mathcal{L}_\infty \triangleq \{\boldsymbol{u} : \sup_{t\in\mathbb{R}^+} \|\boldsymbol{u}(t)\|_p < \infty\}$.

Proposition 1 [26]

Let the function $\boldsymbol{f}(\boldsymbol{x}, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ be uniformly continuous in $\boldsymbol{x}$, uniformly in t, and uniformly continuous in t, uniformly in $\boldsymbol{x}$. Then, given any uniformly continuous function $\boldsymbol{y}(t) : \mathbb{R} \to \mathbb{R}^n$, the function $\boldsymbol{g}(t) \triangleq \boldsymbol{f}(\boldsymbol{y}(t), t)$ is uniformly continuous in t.

Proposition 2 [27]

Let $D \subset \mathbb{R}^n$ be a domain, and let $\boldsymbol{\phi}(\boldsymbol{x}) : D \to \mathbb{R}^m$ be continuous on D. Let $\boldsymbol{x}(t) : \mathbb{R} \to \mathbb{R}^n$ be such that $\boldsymbol{x}(t) \in H$ for $t \geq 0$ where H is a compact subset of D. Additionally, define the set $G \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0}\}$.
Suppose that

$$\lim_{t\to\infty} \boldsymbol{\phi}(\boldsymbol{x}(t)) = \boldsymbol{0}.$$

Then, $\boldsymbol{x}(t)$ approaches the set G as $t \to \infty$.

The following Theorem is adapted from [28] and can be obtained by making appropriate modifications to the proof of Corollary 5.3 in [28, p. 62]. It is to be noted that this theorem is also a special case of Theorem 5.8 in [29, p. 305]

Theorem 8 [28, p. 62]

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}, t) + \boldsymbol{h}(\boldsymbol{x}, t), \tag{68}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$. It is assumed that there is a domain $D_o \subset \mathbb{R}^n$ such that $\boldsymbol{g}(\boldsymbol{x}, t), \boldsymbol{h}(\boldsymbol{x}, t) \in C(D_o \times \mathbb{R})$, and that $\boldsymbol{h}(\boldsymbol{x}, t) \to \boldsymbol{0}$ uniformly in $\boldsymbol{x}$ on compact subsets of D_o as $t \to \infty$. It is further assumed that there exists a $T \in \mathbb{R}$ such that $\boldsymbol{g}(\boldsymbol{x}, t) = \boldsymbol{g}(\boldsymbol{x}, t + T)$ for all $(\boldsymbol{x}, t) \in C(D_o \times \mathbb{R})$.
Suppose that there exists a solution $\boldsymbol{\phi}(t)$ of (68) and a compact subset $D_1 \subset D_o$, such that $\boldsymbol{\phi}(t) \in D_1$ for all $t \geq 0$. Let us denote the positive limit set of $\boldsymbol{\phi}(t)$ by $\boldsymbol{\Lambda}^+$.
Then, for any $\boldsymbol{\zeta} \in \boldsymbol{\Lambda}^+$, there exists a sequence $t_m \to \infty$ with $\boldsymbol{\phi}(t_m) \to \boldsymbol{\zeta}$, a $\bar{t} \in [0, T]$ and a solution $\boldsymbol{\psi}(t)$ of

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{g}(\boldsymbol{x}, t + \bar{t}), \tag{69}$$

with $\boldsymbol{\psi}(0) = \boldsymbol{\zeta}$ such that

$$\boldsymbol{\phi}(t + t_m) \to \boldsymbol{\psi}(t), \text{ as } m \to \infty,$$

uniformly in t on compact subsets of $\mathbb{R}$. Furthermore, $\boldsymbol{\psi}(t) \in D_1$ for all $t \in \mathbb{R}$.

Corollary 1

If all of the conditions in Theorem 8 hold, then the positive limit set $\boldsymbol{\Lambda}^+$ of $\boldsymbol{\phi}(t)$ is invariant under

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{g}(\boldsymbol{x}, t), \tag{70}$$

in the sense that if $\boldsymbol{\zeta} \in \boldsymbol{\Lambda}^+$, then there exists a $\bar{t} \in [0, T]$ such that a solution of (70) with initial condition $\boldsymbol{x}(\bar{t}) = \boldsymbol{\zeta}$ is contained in $\boldsymbol{\Lambda}^+$ for all $t \in \mathbb{R}$.

Proof

Consider $\boldsymbol{\phi}(t)$ and $\boldsymbol{\psi}(t)$ given in Theorem 8. Theorem 8 states that $\boldsymbol{\phi}(t + t_m) \to \boldsymbol{\psi}(t)$, as $m \to \infty$, uniformly in t on compact subsets of $\mathbb{R}$. It follows that $\boldsymbol{\phi}(t + t_m) \to \boldsymbol{\psi}(t)$ as $m \to \infty$, pointwise for all $t \in \mathbb{R}$. Therefore, given any $t \in \mathbb{R}$, we conclude that $\boldsymbol{\phi}(\bar{t}_m) \to \boldsymbol{\psi}(t)$ as $m \to \infty$, for the sequence $\bar{t}_m = t + t_m$, which satisfies $\bar{t}_m \to \infty$ as $m \to \infty$. Therefore, by definition, $\boldsymbol{\psi}(t) \in \boldsymbol{\Lambda}^+$. Now, consider the time-shift $s = t + \bar{t}$. Then, (69) becomes

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{g}(\boldsymbol{x}, s). \tag{71}$$

Since $\boldsymbol{\psi}(t)$ is a solution of (69), with initial condition $\boldsymbol{\psi}(0) = \boldsymbol{\zeta}$, it then follows that $\boldsymbol{x}(s) = \boldsymbol{\psi}(t + \bar{t})$ is a solution of (71), with initial condition $\boldsymbol{x}(\bar{t}) = \boldsymbol{\zeta}$. Since s is a dummy variable, the result follows by replacing s with t in (71). □